LECTURE NOTES: MICROLOCAL METHODS IN DYNAMICAL SYSTEMS

MIHAJLO CEKIĆ

Note. Comments are very welcome and may be sent to mihajlo.cekic@math.uzh.ch.

CONTENTS

Notation 3
1. Summary: Locally Convex Spaces, Distribution Theory, Fourier Analysis 4
1.1. Topological vector spaces 4
1.2. Space of distributions 6
1.3. Schwartz kernel theorem 8
1.4. Fourier transform 10
2. Symbol classes and Oscillatory integrals 13
2.1. Symbol classes 13
2.2. Oscillatory integrals 17
3. Method of stationary phase 23
4. Algebra of pseudodifferential operators 26
4.1. Compositions 26
4.2. Changes of variables 31
5. Elliptic operators and L^2-continuity 33
5.1. Parametrices 33
5.2. L^2-continuity 35
5.3. Sobolev spaces and PDOs 37
6. The wavefront set of a distribution 41
7. Manifolds 45
7.1. Function spaces on manifolds 46
7.2. Pseudodifferential operators on manifolds 46
8. Hyperbolic dynamics 48
8.1. Definitions 48
8.2. Construction of the escape function 50
8.3. Symbols and operators of variable order 54
8.4. Existence of resonances 57
8.5. Meromorphic continuation of operators 61
8.6. Ergodicity 63
8.7. Mixing 64
8.8. Speed of mixing 67
9. Sinai-Ruelle-Bowen measure 67
9.1. Spectral approach to SRB measures 68
9.2. Absolute continuity and SRB measures 80

References 82

Notation

Throughout the notes, we will use the following notation:

(1) For $N \in \mathbb{Z}_{\geq 0}$, the space of N times continuously differentiable functions with support in $K \subset \mathbb{R}^n$ is denoted by $C_0^N(K)$.

(2) For an open set $\Omega \subset \mathbb{R}^n$, denote by $C^\infty(\Omega)$ the space of smooth (infinitely differentiable) functions on Ω; by $C_0^\infty(\Omega)$ the space of compactly supported smooth functions. Similarly $C^N(\Omega)$ denotes the space of N-times continuously differentiable functions on Ω.

(3) Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \big(\mathbb{Z}_{\geq 0}\big)^n$, denote $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Write $D^\alpha := (-i)^{|\alpha|}\partial^\alpha$.

(4) For $p \geq 1$ and $\Omega \subset \mathbb{R}^n$ open, denote by $L^p(\Omega)$ the Banach space of L^p-integrable functions with the norm $\|\phi\|_{L^p}^p = \int_\Omega |\phi|^p(x)dx$. If $p = \infty$, denote by $L^\infty(\Omega)$ the Banach space of bounded (up to measure zero) functions with the norm $\|\phi\|_{L^\infty} = \sup_{x\in\Omega} |\phi|$. Denote by $L^p_{\mathrm{loc}}(\Omega)$ and $L^\infty_{\mathrm{loc}}(\Omega)$ the larger spaces of functions that are L^p and L^∞ over any compact set $K \subset \Omega$, respectively.

(5) Denote $\mathbb{N} := \mathbb{Z}_{\geq 1} = \{1, 2, \dots\}$ the set of natural numbers. $\mathbb{N}_0 := \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.

(6) For $x \in \mathbb{R}^n$ and $\rho > 0$, $\mathbb{B}(x, \rho)$ denotes the open Euclidean ball of radius ρ centred at x.

1. Summary: Locally Convex Spaces, Distribution Theory, Fourier Analysis

Distribution theory makes sense of the ad-hoc constructions coming from physics and provides a framework to study partial differential equations (PDEs). For example, the Dirac delta δ_0 is the distribution that makes sense of the following, for all $f \in C^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \delta_0(x) f(x) dx = f(0).$$

Aim: define the space of *distributions* containing objects such as $\delta(x)$, as the dual space to $C_0^\infty(\mathbb{R}^n)$ with suitable topology, with built-in differentiability properties defined by duality.

1.1. Topological vector spaces.

In what follows, all vectors spaces will be assumed to be complex for definiteness (and the case of real vector spaces follows with obvious changes). A *topological vector space* X is a vector space equipped with a topology in which multiplication with a scalar

$$\mathbb{C} \times X \ni (c, x) \mapsto cx \in X, \qquad X \times X \ni (x, y) \mapsto x + y \in X \tag{1.1}$$

are continuous operations. In fact, since translation is continuous, to specify a topology on X it is enough to specify a base of neighbourhoods at zero.

Moreover, a topological vector space X is called *locally convex* if there is a base of neighbourhoods at zero consisting of sets which are *absorbing*, *balanced*, and *convex*. A set $U \subset X$ is called *absorbing*, if for all $x \in X$ there is a $t > 0$ with $x \in tU = \{ty \mid y \in U\}$; *balanced* if $cx \in U$ for all $x \in U$ and $c \in \mathbb{C}$ with $|c| \leq 1$; and finally *convex* if $tx + (1-t)y \in U$ for all $x, y \in U$ and $0 \leq t \leq 1$. All the topologies we will consider in the course will be locally convex; for simplicity, we will abbreviate a locally convex topological vector space by *LCS*.

Exercise 1.1. *For a topological vector space, any neighbourhood of zero is absorbing. Moreover, there exists a base of open sets at zero which are absorbing and balanced.*

Proof. For the first claim, let $x \in X$ and $0 \in U \subset X$ an open set. By the continuity of multiplication $M : \mathbb{C} \times X \to X$, $M^{-1}(U) \ni (0, x)$ is open. Hence $\exists \delta > 0$ such that $\forall |c| \leq \delta$, $cx \in U$, i.e. U is absorbing.

Similarly, for any U open neighbourhood of zero, $M^{-1}(U) \ni (0, 0)$ is open. Hence there is an open set $V \ni 0$ and $\delta > 0$, such that $cV \subset U$ for $|c| \leq \delta$. This means that $c\delta V \subset U$ for $|c| \leq 1$, and $W := \cup_{|c| \leq 1} c\delta V$ is an absorbing and balanced neighbourhood of zero contained in U. This completes the proof. □

A *seminorm* p on a vector space X is a function $p : X \to \mathbb{R}_{\geq 0}$, *homogeneous* in the sense

$$p(cx) = |c| \cdot p(x) \qquad \forall x \in X, \forall c \in \mathbb{C}$$

and which satisfies the *triangle inequality*

$$p(x + y) \leq p(x) + p(y), \qquad \forall x, y \in X.$$

A family of seminorms $\mathcal{P}$ on a vector space X is called *separating* if for all $x \in X$, there is a $p \in \mathcal{P}$ with $p(x) \neq 0$. Given a separating family $\mathcal{P}$ of seminorms on X, we may induce a locally convex topology on X, by declaring the base of neighbourhoods $\mathcal{B}$ at zero:

$$\mathcal{B} := \left\{\cap_{i=1}^N U(\varepsilon_i, p_i) \mid N \geq 1, \varepsilon_i > 0, p_i \in \mathcal{P}\right\}, \qquad U(\varepsilon, p) := \{x \mid p(x) < \varepsilon\}.$$

Exercise 1.2. *Show that $\mathcal{B}$ indeed induces a LCS topology on X. Call the obtained LCS topology on X the* topology generated by the seminorms in *$\mathcal{P}$.*

Proof. We need to show that the operations in (1.1) are continuous, in order to see that X is a topological vector space. Let us prove it for the addition map M (the continuity of multiplication follows similarly). It suffices to show $M^{-1}(z+U)$ is open for an open set $U = \cap_{i=1}^N U(\varepsilon_i, p_i)$ as above. For that, it suffices to write $z = x + y$ for some $x, y \in X$. Then $(x, y) \in M^{-1}(z+U)$, and we take $V := \cap_{i=1}^N U(\varepsilon_i/2, p_i)$, so by the triangle inequality for p_i we get $(x, y) \in (x+V) \times (y+V) \subset M^{-1}(z+U)$. This completes the proof. □

Example 1.3. *Typical examples of spaces with LCS topology:*

1. *Any normed space;*
2. *For any topological vector space X, we may equip its continuous dual X' (the space of continuous linear functionals) with the topology generated by*
$$\mathcal{P} = \{p_x \mid x \in X\},$$
where $p_x(u) = |u(x)|$. This is known as the weak*-topology *on X.*

Conversely, let X be a locally convex topological vector space. If U is an absorbing, balanced, and convex neighbourhood of zero, introduce the *Minkowski functional* of U
$$p_U(x) = \inf\{t > 0 \mid x \in tU\}.$$

Exercise 1.4. *p_U is a seminorm. Moreover, the topology on X is generated by these seminorms.*

An important special case happens when $\mathcal{P}$ is countable, say $\mathcal{P} = \{p_1, p_2, \dots\}$. Then it can be shown that
$$d(x, y) := \sum_{k=1}^{\infty} 2^{-k} \cdot \frac{p_k(x-y)}{1 + p_k(x-y)}$$
gives a metrisation of the topology on X generated by $\mathcal{P}$. Conversely, one can show that if a locally convex topological vector space is metrisable, then it can be generated by a countable family of seminorms.

A *Fréchet space* is a locally convex topological vector space which is metrisable with a complete metric.

Example 1.5. Given a compact set $K \subset \mathbb{R}^n$, introduce the norms p_N for $N \in \mathbb{Z}_{\geq 0}$ on $C_0^N(K)$ as follows
$$p_{N,K}(\phi) = \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi(x)|.$$

Exercise 1.6. *Show that $C_0^N(K)$ with norm $p_{N,K}$ is a Banach space.*

Similarly, we may topologise $C_0^\infty(K)$ by using the seminorms in the set $\mathcal{P} = \{p_{N,K} \mid N \geq 0\}$. The obtained LCS topology is a Fréchet topology, since $C_0^N(K)$ is a Banach space with the norm $p_{N,K}$, and one has for $(\phi_i)_{i=1}^\infty \subset C_0^\infty(K)$
$$d(\phi_i, \phi_j) \to_{i,j \to \infty} 0 \iff p_{N,K}(\phi_i - \phi_j) \to_{i,j \to \infty} 0, \forall N.$$

If $\Omega \subset \mathbb{R}^n$ is an open set, and $K_j \nearrow \Omega$ is an increasing, exhausting sequence of compact sets, that is, $K_j \subset K_{j+1}$ for $j \in \mathbb{Z}_{\geq 0}$, and $\cup_{j \geq 0} K_j = \Omega$, then we may equip $C^\infty(\Omega)$ with the locally convex topology coming from the family $\mathcal{P} = \{p_{N,K_j} \mid N, j \in \mathbb{Z}_{\geq 0}\}$. Similarly to the previous paragraph, it follows that $C^\infty(\Omega)$ is a Fréchet space with this topology.

We finish the section with a criterion for continuity of maps between two LCS.

Proposition 1.7. *Let X and Y be two LCS and $f : X \to Y$ a linear map between them. Assume $\mathcal{P}$ and $\mathcal{Q}$ are two families of seminorms generating the topologies on X and Y, respectively. Then f is continuous if and only if, for each $q \in \mathcal{Q}$, there exists $p_1, \dots, p_N \in \mathcal{P}$ and a constant $C \geq 0$, such that for all $x \in X$:*

$$q\big(f(x)\big) \leq C \max\big(p_1(x), \dots, p_N(x)\big).$$

Proof. See the proof of [3, Proposition A.1]. □

1.2. **Space of distributions.** Consider an open set $\Omega \subset \mathbb{R}^n$ and the space $C_0^\infty(\Omega)$. We will define a locally convex version of the *inductive limit topology* on $C_0^\infty(\Omega)$ (also known as an *LF-space* structure).

This is done by specifying a base of open sets $\mathcal{B}$ at zero. By definition, $X \in \mathcal{B}$ if

1. X is balanced and convex,
2. for any compact set $K \subset \Omega$, $X \cap C_0^\infty(K)$ is open in the uniform topology defined in Example 1.5.

The sets $X \in \mathcal{B}$ are absorbing by Exercise 1.1, since $C_0^\infty(K)$ are LCS.

Exercise 1.8. *The topology induced by $\mathcal{B}$ on $C_0^\infty(\Omega)$ is LCS.*

Proof. It suffices to check that the operations in (1.1) are continuous. Let us check that for the addition map M (continuity of multiplication is analogous). Let $\phi_i \in C_0^\infty(\Omega)$ for $i = 1, 2, 3$ such that $\phi_1 + \phi_2 = \phi_3$, and $B \in \mathcal{B}$. It suffices to show $M^{-1}(\phi_3 + B)$ contains an open neighbourhood of (ϕ_1, ϕ_2), which we may take to be $W := \big(\phi_1 + \frac{1}{2}B\big) \times \big(\phi_2 + \frac{1}{2}B\big)$. Indeed, by convexity $M(W) \subset \phi_3 + B$ and W is clearly open, completing the proof. □

Next, consider the dual of $C_0^\infty(\Omega)$ denoted by

$$\mathcal{D}'(\Omega) := \Big(C_0^\infty(\Omega)\Big)'.$$

Call the space $\mathcal{D}'(\Omega)$ equipped with the weak*-topology, the space of *distributions* on Ω. Notice that the inclusions $C_0^\infty(K) \subset C_0^\infty(\Omega)$ are continuous by definition; so if $u \in \mathcal{D}'(\Omega)$, then $u|_{C_0^\infty(K)}$ is continuous for all compact K.

Proposition 1.9. *A linear functional $u : C_0^\infty(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$ if and only if for all compact $K \subset \Omega$, there is $N \in \mathbb{Z}_{\geq 0}$ and $C \geq 0$ depending on K, such that for all $\phi \in C_0^\infty(K)$*

$$|\langle u, \phi\rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi|. \tag{1.2}$$

Proof. If $u \in \mathcal{D}'(\Omega)$, then the restriction to $C_0^\infty(K)$ is continuous, so the inequality (1.2) follows from Proposition 1.7.

Conversely, if u is a linear functional satisfying equation (1.2) for all K, by Proposition 1.7 it is continuous on $C_0^\infty(K)$. For $\varepsilon > 0$, if we take an ε-ball $0 \in B_\varepsilon \subset \mathbb{C}$, we easily see that $u^{-1}(B_\varepsilon)$ is convex and balanced, and intersects each $C_0^\infty(K)$ in an open set. Similarly we argue that $u^{-1}(c + B_\varepsilon)$ is open for $c \in \mathbb{C}$ and so u is continuous. □

Note that the Dirac delta δ_0 satisfies the conditions of Proposition 1.9 with $C = 1$ and $N = 0$ and so $\delta_0 \in \mathcal{D}'(\Omega)$.

Remark 1.10. It can be shown that $C_0^\infty(\Omega)$ does not have a countable basis at zero; therefore it is *not metrisable.*

Proposition 1.11. *Let $(\phi_j)_{j=1}^\infty \subset C_0^\infty(\Omega)$. Then $\phi_j \to_{j\to\infty} 0$ in $C_0^\infty(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ with $\operatorname{supp}(\phi_j) \subset K$ for all j, and $\phi_j \to_{j\to\infty} 0$ in $C_0^\infty(K)$.*

Proof. Given that ϕ_j are compactly supported in K and converge to the zero function in $C_0^\infty(K)$, the convergence in the LCS topology of $C_0^\infty(\Omega)$ immediately follows from its definition.

Conversely, suppose for the sake of contradiction that the supports of ϕ_j are contained in compact sets $K_j \nearrow \Omega$ and we assume $K_0 = \emptyset$ for convenience. Find points $x_1 \in K_1$, $x_2 \in K_2 \setminus K_1, \dots, x_j \in K_j \setminus K_{j-1}, \dots$ for $j = 2, \dots,$ such that $\phi_j(x_j) \neq 0$. Introduce, for $\phi \in C_0^\infty(\Omega)$

$$p(\phi) := \sum_{j=1}^\infty \sup\Big\{\Big|\frac{\phi(x)}{\phi_j(x_j)}\Big| \mid x \in K_j \setminus K_{j-1}\Big\}. \tag{1.3}$$

For each compact set $K \subset \Omega$ and $\phi \in C_0^\infty(K)$, $p(\phi)$ is estimated by a constant multiple of $\sup|\phi|$ and so by Proposition 1.9 (by the analogous version valid for seminorm maps $X \to \mathbb{R}$), p is continuous when restricted to $C_0^\infty(K)$. It is then straightforward to check (using the triangle inequality and the homogeneity of p) that p is continuous on $C_0^\infty(\Omega)$ (by the definition of the topology on $C_0^\infty(\Omega)$). Hence $p(\phi_j) \to 0$ as $j \to \infty$ by assumption; but by construction $p(\phi_j) \geq 1$, contradiction.

Therefore there is a K with $\operatorname{supp}(\phi_j) \subset K$ for all j, and the main claim easily follows. ☐

Now given a partial derivative ∂^α, we may extend its action to distributions via duality

$$\langle \partial^\alpha u, \phi\rangle := \langle u, (-1)^{|\alpha|}\partial^\alpha \phi\rangle, \qquad \forall u \in \mathcal{D}'(\Omega), \forall \phi \in C_0^\infty(\Omega).$$

Similarly, extend the multiplication by a function $f \in C^\infty(\Omega)$ as

$$\langle fu, \phi\rangle := \langle u, f\phi\rangle.$$

---

end of Lecture 1, 20.09.2022

Given a linear partial differential operator $P = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$ of degree m, where $a_\alpha \in C^\infty(\Omega)$, we extend its action to distributions via

$$\langle Pu, \phi\rangle := \langle u, {}^tP\phi\rangle.$$

Here ${}^tP$ is the *adjoint* of P, given by

$${}^tP\phi = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha(a_\alpha \phi). \tag{1.4}$$

Exercise 1.12. *Show $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ and $P : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ are continuous linear maps, and that ${}^t({}^tP) = P$.*

Proof. For both of the continuity statements, it suffices to consider the case $P = \partial^\alpha$ and $P = M_f$, where M_f is the multiplication operator by a function $f \in C^\infty(\Omega)$. Let us consider only the latter case (the former is follows analogously). By the definition of the weak*-topology, the map $M_f : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is continuous if and only if for each $\phi \in C_0^\infty(\Omega)$, $\mathcal{D}'(\Omega) \ni u \mapsto |\langle u, f\phi\rangle| \in \mathbb{R}$ is continuous, which again follows from the definition.

Next, we see that $M_f : C_0^\infty(K) \to C_0^\infty(K)$ is continuous for each $K \subset \Omega$ compact (one needs to use the Leibniz rule). Using the definition of the topology on $C_0^\infty(\Omega)$, it is then easy to show that $M_f^{-1}(B) \in \mathcal{B}$ is open for $B \in \mathcal{B}$, and the continuity follows by using the linearity of M_f.

The final claim follows directly by linearity and checking the cases $P = \partial^\alpha$ and $P = M_f$. ☐

Example 1.13. *The following are distributions:*

1. *Any* $f \in L^1_{\mathrm{loc}}(\Omega)$ *defines a distribution by*
$$\langle f, \phi \rangle := \int_\Omega f(x)\phi(x)dx, \qquad \phi \in C_0^\infty(\Omega).$$
In particular we have $C^\infty(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$.
2. *For a hypersurface* $S \subset \mathbb{R}^n$, $\delta_S(\phi) := \int_S \phi$ *is the Dirac distribution on* S. *(If* $S = \{0\}$ *is a point, then we recover the Dirac delta* δ_0.*)*
3. Principal value distribution. *For* $\phi \in C_0^\infty(\mathbb{R})$, *this is defined as:*
$$\mathrm{p.v.}\left(\frac{1}{x}\right)(\phi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{u(x)}{x} dx.$$
It is an exercise to show that $\mathrm{p.v.}\left(\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R})$ *(such that one may take* $N = 1$ *in* (1.2)*).*

Note that we may *restrict* the distribution $u \in \mathcal{D}'(\Omega)$ to any open subset $U \subset \Omega$ and obtain a distribution in $u|_U \in \mathcal{D}'(U)$ (simply set $\langle u|_U, \phi\rangle := \langle u, \phi \rangle$ for all $\phi \in C_0^\infty(U)$). Say that $u = 0$ on U if the restriction $u|_U$ is zero. Define the *support*, $\mathrm{supp}(u)$, of the distribution u as the complement of the set
$$\{x \in \Omega \mid u = 0 \text{ in a neighbourhood of } x\}.$$
Say that u has *compact support* if $\mathrm{supp}(u) \subset \Omega$ is compact and write $u \in \mathcal{E}'(\Omega)$.

Proposition 1.14. *The distributions in* $\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega)$ *may be extended to continuous linear functionals on* $C^\infty(\Omega)$. *Moreover, we have an identification* $\mathcal{E}'(\Omega) = \big(C^\infty(\Omega)\big)'$.

Proof. Given $u \in \mathcal{E}'(\Omega)$, pick $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $\mathrm{supp}(u)$. For $\phi \in C^\infty(\Omega)$, define $\langle u, \phi\rangle := \langle u, \chi \cdot \phi\rangle$; this is clearly independent of the choice of χ. In fact, u is continuous on $C^\infty(\Omega)$ by Propositions 1.7 and 1.9 (recall the LCS topology on $C^\infty(\Omega)$ was introduced in Example 1.5).

Conversely, by Proposition 1.7 for any continuous linear functional $u : C^\infty(\Omega) \to \mathbb{C}$, there exist N and a compact set $K \subset \Omega$ such that $|\langle u, \phi\rangle| \leq Cp_{N,K}(\phi)$, which shows in particular that $\mathrm{supp}(u)$ is compact and that $u \in \mathcal{E}'(\Omega)$. □

We may also define the *singular support* of $u \in \mathcal{D}'(\Omega)$, $\mathrm{sing\,supp}(u) \subset \Omega$, to be the complement of the largest open set $U \subset \Omega$ such that $u|_U \in C^\infty(U)$, i.e. $u|_U$ is identified with a smooth function (see Example 1.13, 1.). For instance, $\mathrm{sing\,supp}(\delta_0) = \{0\}$, and in general:
$$\mathrm{sing\,supp}(u) \subset \mathrm{supp}(u).$$

1.3. **Schwartz kernel theorem.** Let us consider integral transforms of the form
$$f \mapsto kf := \int_Y k(x,y)f(y)dy$$
where k is called the *kernel of the transform*, a (singular) function on $X \times Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are open sets. Clearly k maps functions on Y to functions on X.

Write $\phi \otimes \psi(x,y) := \phi(x)\psi(y)$ for the tensor product of functions $\phi \in C^\infty(X)$ and $\psi \in C^\infty(Y)$. Given $k \in \mathcal{D}'(X \times Y)$, we may consider the following bilinear form:
$$C_0^\infty(X) \times C_0^\infty(Y) \ni (\phi, \psi) \mapsto \langle k, \phi \otimes \psi\rangle \in \mathbb{C}. \tag{1.5}$$

For fixed ψ we get a linear functional $\langle k\psi, \phi\rangle := \langle k, \phi \otimes \psi\rangle$ on $C_0^\infty(X)$, and similarly for fixed ϕ, we get a linear functional on $C_0^\infty(Y)$.

Exercise 1.15. *For each $\psi \in C_0^\infty(Y)$, $k\psi$ is a distribution. Moreover, the following map is sequentially continuous:*

$$k : C_0^\infty(Y) \ni \psi \mapsto k\psi \in \mathcal{D}'(X). \tag{1.6}$$

Proof. For the first claim, since k is a distribution, by Proposition 1.9 we deduce that $k\psi$ is a distribution.

For the second claim, if $(\psi)_{i=1}^\infty \subset C_0^\infty(Y)$ is such that $\psi_i \to \psi$ in $C_0^\infty(Y)$, then for any $\phi \in C_0^\infty(X)$ we have $\phi \otimes \psi_i \to \phi \otimes \psi$ in $C_0^\infty(X \times Y)$; here we use the interpretation of sequential convergence proved in Proposition 1.11. Thus $k\psi_i \to k\psi$ as $i \to \infty$, completing the proof. □

Say that the map in (1.6) is generated by a *Schwartz kernel* k. The main point of this section is the following theorem, known as the *Schwartz kernel theorem*, which associates a kernel to a sequentially continuous map in (1.6):

Theorem 1.16 (Schwartz kernel theorem)**.** *A linear map $\mu : C_0^\infty(Y) \to \mathcal{D}'(X)$ is sequentially continuous if and only if it is generated by a Schwartz kernel $k \in \mathcal{D}'(X \times Y)$, such that:*

$$\langle \mu\psi, \phi\rangle = \langle k, \phi \otimes \psi\rangle, \qquad \forall \phi \in C_0^\infty(X), \forall \psi \in C_0^\infty(Y).$$

Moreover, the kernel k is uniquely determined by μ.

Proof. See the proof of [3, Theorem 6.1.1]. (Note that the uniqueness follows from the density of $C_0^\infty(X) \otimes C_0^\infty(Y) \subset C_0^\infty(X \times Y)$.) □

By swapping x and y, we obtain the *transposed kernel* ${}^t k \in \mathcal{D}'(Y \times X)$, by requiring that

$$\langle {}^t k, \chi\rangle = \langle k, {}^t \chi\rangle, \qquad \forall \chi \in C_0^\infty(Y \times X),$$

where ${}^t\chi(x, y) = \chi(y, x)$. Call a kernel $k \in \mathcal{D}'(X \times Y)$ a *regular kernel* if it has the following mapping properties:

$$k : C_0^\infty(Y) \to C^\infty(X), \qquad {}^t k : C_0^\infty(X) \to C^\infty(Y).$$

We can then extend k and ${}^t k$ to continuous maps $k : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ and ${}^t k : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ by duality:

$$\langle kv, \phi\rangle := \langle v, {}^t k\phi\rangle, \qquad \langle {}^t ku, \psi\rangle := \langle u, k\psi\rangle \tag{1.7}$$

for all $u \in \mathcal{E}'(X)$ and $v \in \mathcal{E}'(Y)$, and test functions $\phi \in C_0^\infty(X)$ and $\psi \in C_0^\infty(Y)$. If we equip $\mathcal{E}'(\Omega)$ with the weak*-topology of $\big(C^\infty(\Omega)\big)'$, by the definition of weak*-topologies, k and ${}^t k$ are continuous. We will often meet regular kernels in the course (e.g. the Schwartz kernel of a pseudodifferential operator).

Exercise 1.17 (Differential operators)**.** *Consider $P : C_0^\infty(X) \to C_0^\infty(X)$ a differential operator with C^∞ coefficients. Then the Schwartz kernel of this operator is given by*

$$k_P(x, y) = {}^t P(y, \partial_y)\delta(x - y),$$

where $\delta(x - y) \in \mathcal{D}'(X \times X)$ (equivalently, $\delta(x - y) = \delta_\Delta$ is the Dirac distribution on Δ, where $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ is the diagonal) is the Schwartz kernel of the identity map $P = \mathrm{Id}$, defined via

$$\langle k_{\mathrm{Id}}, \chi\rangle := \int_X \chi(x, x) dx, \qquad \chi \in C_0^\infty(X \times X).$$

Proof. Note that P is continuous by Exercise 1.12. The case of $P = \mathrm{Id}$ follows from the computation, valid for $\phi, \psi \in C_0^\infty(X)$:

$$\langle k_{\mathrm{Id}}, \phi \otimes \psi \rangle = \langle \phi, \psi \rangle = \int_X \phi(x)\psi(x)dx = \langle \delta(x-y), \phi \otimes \psi \rangle.$$

Next, the general case follows from the computation:

$$\langle {}^tP(y, \partial_y)\delta(x-y), \phi \otimes \psi \rangle = \langle \delta(x-y), \phi \otimes P\psi \rangle = \int_X \phi(x)P\psi(x)dx = \langle P\psi, \phi \rangle.$$

□

1.4. **Fourier transform.** In this section we recall the definition and the basic properties of the Fourier transform.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by, for $\xi \in \mathbb{R}^n$:

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx. \tag{1.8}$$

Here $x \cdot \xi = \sum_{j=1}^n x_j\xi_j$. It is an immediate consequence of the Dominated convergence theorem that $f \in C(\mathbb{R}^n)$ for $f \in L^1(\mathbb{R}^n)$.

Note that we cannot use duality directly to extend the Fourier transform to $\mathcal{D}'(\mathbb{R}^n)$, since $\mathcal{F}$ does not map $C_0^\infty(\mathbb{R}^n)$ to itself. We therefore introduce the space $\mathcal{S}(\mathbb{R}^n)$ of *Schwartz functions* or *rapidly decreasing functions*, by declaring $\phi \in \mathcal{S}(\mathbb{R}^n)$ if ϕ is smooth and

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi| < \infty, \qquad \forall \alpha, \beta \in (\mathbb{Z}_{\geq 0})^n, \tag{1.9}$$

where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. The definition (1.9) means that every derivative $\partial^\beta\phi$ is decaying faster than any polynomial. In fact, we have:

Exercise 1.18. *The seminorms $\|\bullet\|_{\alpha,\beta}$ defined in (1.9) above generate an LCS topology on $\mathcal{S}(\mathbb{R}^n)$ that is Fréchet.*

Let us collect a few basic properties of Fourier transform in one claim:

Proposition 1.19. *Let $f, g \in L^1(\mathbb{R}^n)$. Then:*

1. $|\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R}^n)}$ *and* $\hat{f}(\eta) \to 0$ *as* $|\eta| \to \infty$;
2. $\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(x)g(x)dx$;
3. *The* convolution $f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$ *exists for almost all* $x \in \mathbb{R}^n$. *Also,* $f * g \in L^1(\mathbb{R}^n)$, *and* $\widehat{f * g} = \hat{f}\hat{g}$;
4. $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ *continuously;*
5. $\widehat{D^\alpha\phi}(\xi) = \xi^\alpha\hat{\phi}(\xi)$ *and* $\widehat{x^\alpha\phi} = (-1)^{|\alpha|}D^\alpha\hat{\phi}(\xi)$ *for* $\phi \in \mathcal{S}(\mathbb{R}^n)$;
6. $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is a continuous isomorphism and its inverse is given by*
$$\phi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi}(\xi)e^{ix\cdot\xi}d\xi. \tag{1.10}$$
Hence, $\mathcal{F}$ extends by duality to an isomorphism on the space of Schwartz distributions, *$\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, where $\langle \mathcal{F}(u), \phi \rangle := \langle u, \mathcal{F}(\phi) \rangle$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. (By definition $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$.)*

7. *The Fourier transform extends continuously to an isomorphism $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, such that $\|\mathcal{F}f\|_{L^2} = (2\pi)^{\frac{n}{2}}\|f\|_{L^2}$ for any $f \in L^2(\mathbb{R}^n)$.*

Proof. 1. The first inequality follows directly from (1.8). For the second property (known as the Riemann-Lebesgue Lemma), observe firstly that it is true for the characteristic function $f = \chi_{\prod_{i=1}^n (a_i, b_i)}$ of the rectangle $\prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$, by an explicit computation. Hence it is also true for any linear combination of characteristic functions. Finally, it suffices to use the density of step functions in $L^1(\mathbb{R}^n)$.

2. By 1., $f\hat{g}, \hat{f}g \in L^1(\mathbb{R}^n)$. Therefore by Fubini's theorem:

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{-ix\cdot\xi} g(\xi)d\xi = \int_{\mathbb{R}^n} g(\xi) \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx = \int_{\mathbb{R}^n} \hat{f}(x)g(x)dx.$$

This is sometimes known as Parseval's identity.

3. Note that $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)|dxdy = \|f\|_{L^1}\|g\|_{L^1}$, so by Fubini's theorem $f * g(x)$ exists almost everywhere, and $f * g \in L^1(\mathbb{R}^n)$. Again by Fubini's theorem:

$$\widehat{f * g}(\xi) = \int_{\mathbb{R}^n} g(y)dy \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-y)dx = \int_{\mathbb{R}^n} g(y)dy \int_{\mathbb{R}^n} e^{-iy\cdot\xi}e^{-iz\cdot\xi} f(z)dz = \hat{f}(\xi)\hat{g}(\xi),$$

where in the second equality we used the substitution $x = y + z$. This completes the proof.

4. Observe that for $N > n/2$, we have $(1 + |x|^2)^{-N} \in L^1(\mathbb{R}^n)$. Now for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have for some $C > 0$:

$$(1 + |x|^2)^N |\phi(x)| \leq C \sum_{|\alpha|\leq 2N} \|\phi\|_{\alpha,0}, \qquad \forall x \in \mathbb{R}^n.$$

It follows that with a possibly different $C > 0$, $\|\phi\|_{L^1} \leq C\sum_{|\alpha|\leq 2N} \|\phi\|_{\alpha,0}$, which completes the proof by using Proposition 1.7.

5. By 4., and since $\mathcal{S}(\mathbb{R}^n)$ is invariant under differentiation and multiplication by polynomials (easy to check from the definition (1.9) of seminorms), these Fourier transforms exist. In fact (by induction), it suffices to prove the claim for $|\alpha| = 1$. We have for $k \in \{1, \dots, n\}$:

$$\widehat{D_k\phi}(\xi) = \int_{\mathbb{R}^n} D_k\phi(x)e^{-ix\cdot\xi}dx = \int_{\mathbb{R}^n} (D_k(\phi e^{-ix\cdot\xi}) + \phi \cdot \xi_k e^{-ix\cdot\xi})dx = \xi_k\widehat{\phi}(\xi),$$

where in the third equality the first term integrates to zero by the decaying property of $\phi \in \mathcal{S}(\mathbb{R}^n)$. The other claim follows similarly.

6. To show the continuous mapping property, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n$, we have:

$$\|\hat{\phi}\|_{\alpha,\beta} = \|(-1)^{|\beta|}\widehat{D^\alpha(x^\beta\phi)}\|_{0,0} \leq \|D^\alpha(x^\beta\phi)\|_{L^1},$$

where we used 5. in the first equality and 1. in the last equality. Thus the map is continuous by Proposition 1.7.

Next, we show (1.10). To start with, we claim that the Fourier transform of a Gaussian is:

$$\mathcal{F}\left(e^{-\frac{1}{2}|x|^2}\right) = (2\pi)^{\frac{n}{2}}e^{-\frac{1}{2}|\xi|^2}. \tag{1.11}$$

To see (1.11), observe that $\mathcal{F}\left(e^{-\frac{1}{2}|x|^2}\right) = \prod_{i=1}^n \mathcal{F}_{x_i\to\xi_i}\left(e^{-\frac{1}{2}x_i^2}\right)$, so it suffices to show the case $n=1$. Set $\phi := e^{-\frac{1}{2}x^2} \in \mathcal{S}(\mathbb{R})$ and observe

$$(iD_x + x)\phi = 0 \implies (iD_\xi + \xi)\hat{\phi} = 0,$$

where we used 5. for the implication. So by uniqueness of solution to ODE,

$$\hat{\phi}(\xi) = \sqrt{2\pi}\cdot e^{-\frac{1}{2}\xi^2}, \tag{1.12}$$

by the classical formula $\sqrt{2\pi} = \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}dx$.

Now let $\varepsilon > 0$, and apply 2. to $f := e^{-\frac{1}{2}\varepsilon^2|x|^2}$ and $g := \phi$, to get:

$$\int_{\mathbb{R}^n} \hat{\phi}(\xi)e^{-\frac{1}{2}\varepsilon^2|\xi|^2}d\xi = (2\pi)^{\frac{n}{2}}\varepsilon^{-n}\int_{\mathbb{R}^n}\phi(x)e^{-\frac{1}{2\varepsilon^2}|x|^2}dx = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\phi(\varepsilon x)e^{-\frac{1}{2}|x|^2}dx,$$

where in the first equality we used (1.11) (the re-scaled version) and in the second equality we used the substitution $z := \frac{x}{\varepsilon}$. By the Dominated convergence theorem applied to the two sides of the previous equality:

$$\int_{\mathbb{R}^n}\hat{\phi}(\xi)d\xi = (2\pi)^{\frac{n}{2}}\phi(0)\int_{\mathbb{R}^n}e^{-\frac{1}{2}|x|^2}dx = (2\pi)^n\phi(0).$$

This shows (1.10) for $x = 0$; for arbitrary $x \in \mathbb{R}^n$, it suffices to observe that the translation $\tau_x\phi(y) := \phi(y+x) \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\widehat{\tau_x\phi}(\xi) = e^{ix\cdot\xi}\hat{\phi}(\xi)$, and apply the previous result to $\tau_x\phi$. This completes the proof of (1.10).

Now (1.10) shows injectivity of $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ directly, and the surjectivity also follows by taking $\psi(\xi) := (2\pi)^{-n}\hat{\phi}(-\xi)$ and observing $\hat{\psi} = \phi$. Continuity of the inverse follows similarly.

7. Let us first take $\phi \in \mathcal{S}(\mathbb{R}^n)$; then applying 2. to $f := \phi$ and $g := \overline{\hat{f}}$, we get

$$\|\hat{\phi}\|_{L^2}^2 = \int_{\mathbb{R}^n}|\hat{\phi}(\xi)|^2d\xi = \int_{\mathbb{R}^n}\phi(x)\hat{\overline{\hat{\phi}}}(x)dx = (2\pi)^n\|\phi\|_{L^2}^2,$$

where in the second equality we used $\overline{\hat{\phi}}(\xi) = \hat{\overline{\phi}}(-\xi)$ and (1.10) applied to $\overline{\phi}$ (so $\hat{\hat{\overline{\phi}}} = (2\pi)^n\overline{\phi}$).

Now, since $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is dense (see Exercise 1.20 below), $\mathcal{F}$ admits a continuous extension to $L^2(\mathbb{R}^n)$, mapping to $L^2(\mathbb{R}^n)$. Similarly (using (1.10)), it is invertible. $\square$

Exercise 1.20. *$C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is dense.*

Proof sketch. Given $f \in L^2(\mathbb{R}^n)$, then it is easy to see $\chi_{B_i}f \to f$ in $L^2(\mathbb{R}^n)$ as $i \to \infty$, where χ_{B_i} is the indicator function of the ball $B_i \subset \mathbb{R}^n$ of radius i. Therefore it suffices to consider $f \in L^2(\mathbb{R}^n)$ of compact support. Now, let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a smooth cut-off function such that $\int_{\mathbb{R}^n}\phi(x)dx = 1$ (also called a *mollifier*). Then set $\phi_k := k^n\phi(kx) \in C_0^\infty(\mathbb{R}^n)$. It is then an exercise to check that 1) the convolution $\phi_k * f \in C_0^\infty(\mathbb{R}^n)$, and 2) $\phi_k * f \to f$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. (The proof is similar to [3, Theorem 1.2.1]. The whole process is called *mollification* or *regularisation* of f, and applies to more general situations.) $\square$

2. Symbol classes and Oscillatory integrals

In this section, we introduce *symbol classes*, following the first chapters of [4,5], that we will use to define distributions called *oscillatory integrals*, which in turn will serve as Schwartz kernels of *pseudodifferential operators*.

Motivation and aim. Given a differential operator $P = \sum_{|\alpha|\leq N} a_\alpha D^\alpha$ with constant coefficients $a_\alpha \in \mathbb{C}$, we observe (by Proposition 1.19, 5.) that for $\phi \in C_0^\infty(\mathbb{R}^n)$:

$$P\phi = \mathcal{F}^{-1}(p\mathcal{F}\phi), \qquad p = \sum_{|\alpha|\leq N} a_\alpha \xi^\alpha.$$

Our goal is to define more general p (non-polynomial and depending on x), called *symbols*, for which we will get more general (pseudo)operators P, that will form a *calculus*. For instance, if $p = \frac{1}{1+|\xi|^2}$ we get:

$$(1-\Delta)^{-1} = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^2}\mathcal{F}\right),$$

where $-\Delta := -\sum_{i=1}^n \partial_{x_i}^2$ is the Laplacian, and this operator defines an inverse to $(1-\Delta)$, a non-differential operator.

end of Lecture 2, 27.09.2022

More generally, for arbitrary $p(x,\xi)$ and a test function ϕ we formally have (by Proposition 1.19)

$$P\phi(x) = \mathcal{F}^{-1}p\mathcal{F}\phi(x) = \int_{y\in\mathbb{R}^n}\phi(y)dy\,\underbrace{(2\pi)^{-n}\int_{\xi\in\mathbb{R}^n} e^{i(x-y)\cdot\xi}p(x,\xi)d\xi}_{K(x,y):=}. \tag{2.1}$$

Then K is the Schwartz kernel of P, and we would like to understand for which p is K a well-defined distribution.

2.1. Symbol classes.

Let $X\subset\mathbb{R}^n$ be an open set and $N\in\mathbb{N}$. In the following definition take $0\leq\rho\leq 1$, $0\leq\delta\leq 1$.

Definition 2.1 (Symbol classes)**.** *Define $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ as the space of all $a\in C^\infty(X\times\mathbb{R}^N)$, such that for all compact $K\subset X$ and all multi-indices $\alpha\in\mathbb{N}_0^n$, $\beta\in\mathbb{N}_0^N$, there is a constant $C = C_{K,\alpha,\beta}(a)$ such that:*

$$|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}, \qquad \forall (x,\theta)\in K\times\mathbb{R}^N. \tag{2.2}$$

Say that $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ is the space of all symbols of order m *and of* type (ρ,δ)*. If $\rho = 1$ and $\delta = 0$, use the notation $S^m(X\times\mathbb{R}^N) := S^m_{1,0}(X\times\mathbb{R}^N)$.*

Remark 2.2. We will be almost exclusively interested in the case $\rho = 1$ and $\delta = 0$, that is, the symbol class $S^m(X\times\mathbb{R}^N)$. Most of the results of this section will however be valid more generally for arbitrary (ρ,δ) as above.

The space of symbols of order $-\infty$ is defined by

$$S^{-\infty}(X\times\mathbb{R}^N) := \bigcap_{m\in\mathbb{R}} S^m(X\times\mathbb{R}^N). \tag{2.3}$$

Equivalently, it follows from the definition that symbols a of order $-\infty$ are such that for all compact sets $K\subset X$, multi-indices $\alpha\in\mathbb{N}_0^n$ and $\beta\in\mathbb{N}_0^N$, $M\in\mathbb{R}$, there is a $C>0$ such that:

$$|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{-M}, \qquad (x,\theta)\in K\times\mathbb{R}^N. \tag{2.4}$$

Exercise 2.3. We give a few examples of symbols:

1. If $a_\alpha \in C^\infty(X)$, then $a(x,\theta) = \sum_{|\alpha|\leq m} a_\alpha(x)\theta^\alpha$ belongs to $S^m(X \times \mathbb{R}^N)$.
2. More generally, if $a \in C^\infty(X \times \mathbb{R}^N)$ is *positively homogeneous of degree* m (or *positively* m*-homogeneous*) for $|\theta| \geq 1$, i.e. $a(x, \lambda\theta) = \lambda^m a(x,\theta)$ for $\lambda \geq 1$ and $|\theta| \geq 1$, then $a \in S^m(X \times \mathbb{R}^N)$.

   Proof. Let $K \subset X$ be compact. For instance, if α and β are zero, then if $C := \sup_{x\in K, |\theta|\leq 1} |a(x,\theta)|$ it holds
   $$\frac{|a(x,\theta)|}{(1+|\theta|)^m} \leq C,$$
   where the inequality is obvious for $|\theta| \leq 1$, and for $|\theta| \geq 1$, we use the homogeneity of a. In the general case, by the chain rule it follows that
   $$\partial_\theta^\beta a(x, \lambda\theta) = \lambda^{m-|\beta|}\partial_\theta^\beta a(x,\theta), \qquad \lambda \geq 1, |\theta| \geq 1, x \in X,$$
   i.e. $\partial_\theta^\beta a$ and so $\partial_x^\alpha \partial_\theta^\beta a$ are positively $(m - |\beta|)$-homogeneous for $|\theta| \geq 1$ and $\lambda \geq 1$. The conclusion follows. □
3. The symbol $e^{-|\theta|^2} \in S^{-\infty}(X \times \mathbb{R}^N)$.
4. We have $e^{ix\cdot\xi} \in S^0_{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$, but $e^{ix\cdot\xi} \notin S^m(\mathbb{R}^n \times \mathbb{R}^n)$ for any $m \in \mathbb{R}$.

   Proof. Compute the derivatives
   $$\partial_x^\alpha \partial_\xi^\beta e^{ix\cdot\xi} = i^{|\alpha|+|\beta|}\xi^\alpha x^\beta e^{ix\cdot\xi}.$$
   It is evident that for x in each compact set $K \subset \mathbb{R}^n$, this satisfies the estimates (2.2) for $\rho = 0$ and $\delta = 1$. On the other hand, $a \notin S^m(\mathbb{R}^n \times \mathbb{R}^n)$ for any m, since applying ∂_x^α produces powers of ξ in the derivative, contradicting (2.2) for $\rho = 1$ and $\delta = 0$. □
5. Exotic: assume $f \in C^\infty(X \times \mathbb{R}^N; [0,\infty))$ and is positively homogeneous of degree 1 for $|\theta| \geq 1$. Then $e^{-f} \in S^0_{\frac{1}{2},\frac{1}{2}}(X \times \mathbb{R}^N)$.

   Proof. See [4, Example 1.4]. □

Remark 2.4. There is little point in introducing $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ for $\rho > 1$ (or $\delta < 0$). For instance, if $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ for $m < 0$ and $\rho > 1$, equation (2.2) yields that after applying the ∂_θ^β derivative many times and then integrating, that there is a gain in the decay in $\partial_\theta^{|\beta|-1}$ derivative. Iterating, we obtain $a \in S^{-\infty}(X \times \mathbb{R}^N)$.

Equip the space $S^m(X \times \mathbb{R}^N)$ with the LCS topology given by the seminorms, for each compact set $K \subset X$ and multi-indices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^N$:
$$P_{K,\alpha,\beta}(a) = \sup_{(x,\theta)\in K\times\mathbb{R}^N} \frac{|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)|}{(1+|\theta|)^{m-|\beta|}}, \qquad a \in S^m(X \times \mathbb{R}^N). \tag{2.5}$$

Exercise 2.5. *Let $K_j \nearrow X$ be an exhausting sequence of sets. Then the family $\mathcal{P} := \{P_{K_j,\alpha,\beta} \mid j \geq 1, \alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^N\}$ generates an LCS topology in $S^m(X \times \mathbb{R}^N)$ which is Fréchet. Similarly, the space $S^{-\infty}(X \times \mathbb{R}^N)$ carries the natural Fréchet topology, by choosing "the best" constants in* (2.4).

Proof. We prove the first claim only and the second one follows similarly. Assume $(a_i)_{i=1}^\infty$ is a Cauchy sequence in $S^m(X \times \mathbb{R}^N)$. Equivalently, $p_{K,\alpha,\beta}(a_i - a_j) \to 0$ as $i, j \to \infty$, for all compact sets $K \subset X$ and all multi-indices α and β. In particular, this implies that $(a_i)_{i=1}^\infty$ is a Cauchy sequence in $C^\infty(K \times \{|\theta| \leq L\})$ for any $L > 0$, i.e. it is Cauchy in $C^\infty(X \times \mathbb{R}^N)$. Since the latter is a Fréchet space (see Exercise 1.6), we conclude that there exists $a \in C^\infty(X \times \mathbb{R}^N)$ such that $a_j \to a$ in $C^\infty(X \times \mathbb{R}^N)$. Next, being Cauchy in $S^m(X \times \mathbb{R}^N)$ implies boundedness (i.e. each seminorms is bounded on the sequence), and it is then straightforward to show $a \in S^m(X \times \mathbb{R}^N)$.

To show convergence in $S^m(X \times \mathbb{R}^N)$, let $\varepsilon > 0$. Then for $i, j > M$ for some large M, $p_{K,\alpha,\beta}(a_i - a_j) < \varepsilon$. Therefore, for any $L > 0$ and $j > N$:

$$\sup_{K \times \{|\theta| \leq L\}} \frac{\partial_x^\alpha \partial_\theta^\beta (a - a_i)}{(1 + |\theta|)^{m - |\beta|}} \leq \sup_{K \times \{|\theta| \leq L\}} \frac{\partial_x^\alpha \partial_\theta^\beta (a - a_j)}{(1 + |\theta|)^{m - |\beta|}} + \sup_{K \times \{|\theta| \leq L\}} \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a_i)}{(1 + |\theta|)^{m - |\beta|}}.$$

By the convergence in $C^\infty(X \times \mathbb{R}^N)$ the first term on the right hand side can be made as smaller than ε, and the second term is smaller than ε. This shows $a_j \to a$ in $S^m(X \times \mathbb{R}^N)$ and completes the proof. $\square$

Next, observe that $\partial_x^\alpha \partial_\theta^\beta : S^m(X \times \mathbb{R}^N) \to S^{m-|\beta|}(X \times \mathbb{R}^N)$ is continuous (using Proposition 1.7), and that $S^m(X \times \mathbb{R}^N) \subset S^{m'}(X \times \mathbb{R}^N)$ if $m \leq m'$.

Let us now prove a few basic properties about the topologies on the spaces of symbols, with the aim to make sense of an asymptotic formula $a \sim \sum_{j=0}^\infty a_j$ where $a_j \in S^{m_k}(X \times \mathbb{R}^N)$ for $m_j \to -\infty$ as $j \to \infty$.

Proposition 2.6. *Assume $m_1, m_2 \in \mathbb{R}$. Then the bilinear map $(a, b) \mapsto ab$*

$$S^{m_1}(X \times \mathbb{R}^N) \times S^{m_2}(X \times \mathbb{R}^N) \to S^{m_1+m_2}(X \times \mathbb{R}^N)$$

is well-defined and continuous.

Proof. If we apply the Leibniz' formula to $\partial_x^\alpha \partial_\theta^\beta a$, we get

$$\partial_x^\alpha \partial_\theta^\beta (ab) = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C(\alpha_1, \alpha_2, \beta_1, \beta_2) \partial_x^{\alpha_1} \partial_\theta^{\beta_1} a \cdot \partial_x^{\alpha_2} \partial_\theta^{\beta_2} b \tag{2.6}$$

for suitable multinomial coefficients $C(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Well-definedness follows from the estimates (2.2) applied to each summand on a set $K \times \mathbb{R}^N$, showing that $\partial_x^\alpha \partial_\theta^\beta (ab) \in S^{m_1+m_2}(X \times \mathbb{R}^N)$. Continuity follows by using (2.6) to estimate the seminorms of $S^{m_1+m_2}(X \times \mathbb{R}^N)$ by sums of products of seminorms on $S^{m_1}(X \times \mathbb{R}^N)$ and $S^{m_2}(X \times \mathbb{R}^N)$, and using an appropriate version of Proposition 1.7 for bilinear maps. $\square$

Next we prove a convergence lemma in the symbol space topology.

Proposition 2.7. *Let $(a_j)_{j=1}^\infty$ be a bounded sequence*[1] *in $S^m(X \times \mathbb{R}^N)$, which converges pointwise for each $(x, \theta) \in X \times \mathbb{R}^N$. Then the pointwise limit a belongs to $S^m(X \times \mathbb{R}^N)$ and for every $m' > m$, we have $a_j \to a$ as $j \to \infty$ in the topology of $S^{m'}(X \times \mathbb{R}^N)$.*

Proof. Step 1. By assumption, the $P_{K,\alpha,\beta}(a_j)$ are bounded uniformly in j. Using this for $|\alpha| + |\beta| = 1$ and Arzelà-Ascoli Theorem, we conclude that $a_j \to a$ uniformly in $C^0(X \times \mathbb{R}^N)$ (i.e. uniformly on compact subsets in the C^0-norm). Let us give the argument to show that $a_j \to a$ uniformly in $C^1(X \times \mathbb{R}^N)$. Next, consider $\partial_{x_1} a_j$ – by the same argument using

[1] A subset $S \subset F$ of an LCS space F is called *bounded* if for each seminorm $p \in \mathcal{P}$ we have p is bounded on S (by a constant depending on p), i.e. $\sup_{s \in S} p(s) < \infty$.

Arzelà-Ascoli Theorem, $\partial_{x_1} a_j \to b$ in $C^0(X \times \mathbb{R}^N)$. In particular, integrating, for some fixed $(x,\theta) \in X \times \mathbb{R}^N$, and for all $\varepsilon > 0$ small enough, we get

$$\frac{1}{\varepsilon}\int_0^\varepsilon b(x + t\mathbf{e}_1, \theta)dt = \frac{a(x+\varepsilon\mathbf{e}_1) - a(x)}{\varepsilon},$$

where $\mathbf{e}_i \in \mathbb{R}^n$ denotes the ith standard coordinate vector. Taking $\varepsilon \to 0$ and by the fundamental theorem of calculus, we conclude $b(x,\theta) = \partial_{x_1} a(x,\theta)$. This argument works similarly ∂_{x_i} and ∂_{θ_i}, and shows that a is continuously differentiable and $a_j \to a$ in $C^1(X \times \mathbb{R}^N)$. Similar (inductive) argument works for higher order derivatives and shows $a_j \to a$ in $C^\infty(X \times \mathbb{R}^N)$.[2] It follows that $a \in S^m(X \times \mathbb{R}^N)$.

Step 2. In order to prove convergence in $S^{m'}(X \times \mathbb{R}^N)$, consider a compact set $K \subset X$ and multi-indices α, β. For each $(x,\theta) \in K \times \mathbb{R}^N$ we have:

$$k_j(x,\theta) := \frac{|\partial_x^\alpha \partial_\theta^\beta (a - a_j)|}{(1+|\theta|)^{m'-|\beta|}} = \frac{1}{(1+|\theta|)^{m'-m}} \frac{|\partial_x^\alpha \partial_\theta^\beta (a - a_j)|}{(1+|\theta|)^{m-|\beta|}}$$

For any $\varepsilon > 0$, we may take L large enough such that for $|\theta| \geq L$, by the boundedness of the second factor, we have $k_j(x,\theta) < \varepsilon$ on $K \times \{|\theta| \geq L\}$. In the regime where $|\theta| \leq L$, by uniform convergence in $C^\infty(X \times \mathbb{R}^N)$ we have $k_j(x,\theta) \to_{j\to\infty} 0$ on $K \times \{|\theta| \leq L\}$. Therefore, combining the two estimates we see that

$$\sup_{(x,\theta) \in K \times \mathbb{R}^N} k_j(x,\theta) \to 0, \qquad j \to \infty,$$

completing the proof. $\square$

Symbols of order $-\infty$ are in fact dense in other finite order symbol spaces.

Proposition 2.8. *If $m' > m$, then $S^{-\infty}(X \times \mathbb{R}^N)$ is dense in $S^m(X \times \mathbb{R}^N)$ in the topology of $S^{m'}(X \times \mathbb{R}^N)$.*

Proof. Let $\chi \in C_0^\infty(X \times \mathbb{R}^N)$ with $\chi = 1$ for $|\theta| \leq 1$ and $\chi = 0$ for $|\theta| \geq 2$. Then define the re-scalings $\chi_j(\theta) := \chi(\frac{\theta}{j})$ for $j \in \mathbb{N}$, which we claim is a bounded sequence in $S^0(X \times \mathbb{R}^N)$. Indeed, we have that

$$\partial_\theta^\alpha \chi_j(\theta) = j^{-|\alpha|} \partial_\theta^\alpha \chi\Big(\frac{\theta}{j}\Big) \begin{cases} = 0, & \text{if } |\theta| \geq 2j \text{ or } |\theta| \leq j, \\ \leq 2^{|\alpha|}\left(1 + \frac{1}{j}\right)^{|\alpha|} \|\partial_\theta^\alpha \chi\|_{L^\infty}(1+|\theta|)^{-|\alpha|}, & \text{otherwise.} \end{cases}$$

(This follows since derivatives of χ are supported on $1 \leq |\theta| \leq 2$.) Let $a \in S^m(X \times \mathbb{R}^N)$ and define $a_j(x,\theta) := \chi_j(\theta) a(x,\theta)$. By Proposition 2.6 we know $a_j \in S^{-\infty}(X \times \mathbb{R}^N)$. Since $a_j \to a$ pointwise, Proposition 2.7 applies and proves the result. $\square$

Now comes one of the main points of this section: we may sum the symbols asymptotically.

[2] Alternatively, conclude that $a_j \to a$ in C^0 by Arzelà-Ascoli as above. Then apply the *interpolation* inequality $|f'(0)| \leq C_\varepsilon(\|f\|_{L^\infty}^{\frac{1}{2}} \|f''\|_{L^\infty}^{\frac{1}{2}} + \|f\|_{L^\infty})$, valid for any $f \in C^2([-\varepsilon,\varepsilon])$. More precisely, argue that $(a_j)_{j=1}^\infty$ is Cauchy in C^1 by applying this inequality to each coordinate of the terms $(a_i - a_j)$, iterate to get Cauchy in C^k for each k, and use Exercise 1.6.

Proposition 2.9. *Let $(m_j)_{j=0}^\infty \subset \mathbb{R}$ such that $m_j \searrow -\infty$ as $j \to \infty$ and let $a_j \in S^{m_j}(X\times\mathbb{R}^N)$. Then there exists $a \in S^{m_0}(X \times \mathbb{R}^N)$, unique modulo $S^{-\infty}(X \times \mathbb{R}^N)$, such that*

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}(X \times \mathbb{R}^N), \qquad \forall k \in \mathbb{N}_0. \tag{2.7}$$

If (2.7) holds, we write $a \sim \sum_{j=0}^\infty a_j$ and call this an *asymptotic sum* or *asymptotic development.*

Proof. Uniqueness part follows from the fact that, if $a' \in S^{m_0}(X \times \mathbb{R}^N)$ also satisfies (2.7), then $a - a' \in S^{m_k}(X \times \mathbb{R}^N)$ for all k and so $a - a' \in S^{-\infty}(X \times \mathbb{R}^N)$.

For the existence, let $\mathcal{P}_j = \{p_{j,0}, p_{j,1}, \dots\}$ be an enumeration of the seminorms defining the topology of $S^{m_j}(X \times \mathbb{R}^N)$, such that we have $p_{j,k} \leq p_{j',k}$ for $j' \geq j$ and all k (this can achieved by taking an exhaustion $K_j \nearrow X$ by compact sets, ordering $\{(K_j, \alpha, \beta) \mid j, \alpha, \beta\} = \{a_1, a_2, \dots\}$ in a sequence, and defining $p_{i,k}$ to be the seminorm (2.2) corresponding to a_i and order m_k). Then Proposition 2.8 implies that, for every $j \in \mathbb{N}_0$, there is $b_j \in S^{-\infty}(X \times \mathbb{R}^N)$ such that for all $0 \leq \mu \leq j-1$

$$P_{j,\mu}(a_j - b_j) \leq 2^{-j}.$$

Therefore, the sum $\sum_{j=k}^\infty (a_j - b_j)$ converges in $S^{m_k}(X \times \mathbb{R}^N)$ for all $k \in \mathbb{N}_0$, since the partial sums $\sum_{j=k}^\ell (a_j - b_j)$ are Cauchy with respect to the seminorms in $\mathcal{P}_k$, or equivalently they are Cauchy with respect to the Fréchet topology on $S^{m_k}(X \times \mathbb{R}^N)$ (see Exercise 2.5). If we set $a := \sum_{j=0}^\infty (a_j - b_j) \in S^{m_0}(X \times \mathbb{R}^N)$, then

$$a - \sum_{j=0}^{k-1} a_j = -\sum_{j=0}^{k-1} b_j + \sum_{j=k}^\infty (a_j - b_j) \in S^{m_k}(X \times \mathbb{R}^N),$$

completing the proof. $\square$

Remark 2.10. A couple of remarks concerning Lemma 2.9:

1. The proposition is reminiscent of a statement known as Borel's lemma, which states that we may construct smooth functions with prescribed jets, i.e. for any given sequence $(a_\alpha)_{\alpha\in\mathbb{N}_0^n}$, there exists $f \in C^\infty(\mathbb{R}^n)$ with $a_\alpha \alpha! = \partial^\alpha f(0)$.
2. The form of the a we constructed can be seen as follows:
$$a = \sum_{j=0}^\infty \Big(1 - \chi\Big(\frac{\theta}{t_j}\Big)\Big) a_j(x,\theta),$$
where $t_j \to \infty$ are chosen to grow sufficiently fast so that the sum and the derivatives converge, and χ as in Proposition 2.8.

---

end of Lecture 3, 04.10.2022

2.2. **Oscillatory integrals.** For $a \in S^m(X \times \mathbb{R}^N)$ and φ a *phase function*, we would like to make sense as distributions of the integrals of the form

$$I(a,\varphi)(x) := \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x,\theta) d\theta. \tag{2.8}$$

Call the integrals of this form *oscillatory integrals*. The kernel (2.1) is an instance of an oscillatory integral (where $\varphi(x,y,\xi) := (x-y)\cdot\xi \in C^\infty(X \times X \times \mathbb{R}^n)$). Next, the conditions that φ needs to satisfy can be formalised in the following definition:

Definition 2.11. *A function $\varphi(x,\theta) \in C^\infty\big(X \times (\mathbb{R}^N \setminus 0)\big)$ is called a* phase function *if for all $(x,\theta) \in X \times (\mathbb{R}^N \setminus 0)$*

1. $\operatorname{Im}\varphi \geq 0$;
2. $\varphi(x,\lambda\theta) = \lambda\varphi(x,\theta)$ *for $\lambda > 0$ (positive homogeneity);*
3. $d_{x,\theta}\varphi = (\partial_{x_1}\varphi, \dots, \partial_{x_n}\varphi, \partial_{\theta_1}\varphi, \dots, \partial_{\theta_N}\varphi) \neq 0$ *(φ has no critical points).*

Remark 2.12. For the applications we have in mind we can safely assume that φ extends to a function in $C^\infty(X \times \mathbb{R}^N)$. (For instance we will often take $\varphi(x,y,\xi) := (x-y)\cdot\xi \in C^\infty(X \times X \times \mathbb{R}^n)$ as discussed above.)

Observe that if $m < -N-k$ for some $k \in \mathbb{N}_0$ the integral in (2.8) converges and $I(a,\varphi) \in C^k(X)$ (use the Dominated convergence theorem for continuity; that this is C^k-regular is obtained in a similar fashion). To make sense of it as an element of $\mathcal{D}'(X)$ we integrate by parts many times to trade x-derivatives for inverse powers of $(1+|\theta|)$.

Lemma 2.13. *Let φ be a phase function. There exist $a_j \in S^0(X \times \mathbb{R}^N)$ and $b_k, c \in S^{-1}(X \times \mathbb{R}^N)$ for $j = 1, \dots, N$ and $k = 1, \dots, n$, such that*

$$L = \sum_{j=1}^N a_j \frac{\partial}{\partial\theta_j} + \sum_{k=1}^n b_k \frac{\partial}{\partial x_k} + c \tag{2.9}$$

satisfies ${}^tLe^{i\varphi} = e^{i\varphi}$. Here ${}^tL$ is the adjoint of L (see (1.4)) and is given explicitly by

$${}^tLu = -\sum_{j=1}^N \frac{\partial(a_j u)}{\partial\theta_j} - \sum_{k=1}^n \frac{\partial(b_k u)}{\partial x_k} + cu.$$

Proof. Introduce

$$\Phi(x,\theta) = \sum_{k=1}^n \Big|\frac{\partial\varphi}{\partial x_k}\Big|^2 + |\theta|^2 \sum_{j=1}^N \Big|\frac{\partial\varphi}{\partial\theta_j}\Big|^2. \tag{2.10}$$

Note that $\Phi(x,\theta) > 0$ for $\theta \neq 0$ since φ has no critical points, $\Phi(x,\lambda\theta) = \lambda^2\Phi(x,\theta)$ for $\lambda > 0$ and $\theta \neq 0$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ a bump function such that $\chi = 1$ near zero. Let

$$\begin{aligned}
{}^tL &:= \frac{1-\chi(\theta)}{i\Phi(x,\theta)}\Big(|\theta|^2 \sum_{j=1}^N \frac{\partial\bar\varphi}{\partial\theta_j}\frac{\partial}{\partial\theta_j} + \sum_{k=1}^n \frac{\partial\bar\varphi}{\partial x_k}\frac{\partial}{\partial x_k}\Big) + \chi(\theta),\\
a_j' &:= \frac{1-\chi(\theta)}{i\Phi(x,\theta)}\cdot|\theta|^2\frac{\partial\bar\varphi}{\partial\theta_j}, \quad b_k' := \frac{1-\chi(\theta)}{i\Phi(x,\theta)}\cdot\frac{\partial\bar\varphi}{\partial x_k}, \quad c' := \chi(\theta).
\end{aligned}$$

Clearly $c' \in S^{-\infty}(X \times \mathbb{R}^N)$. Since the expressions in the definition of a_j' and b_k' are 0 and -1-homogeneous for large $|\theta|$, respectively, by Exercise 2.3, item 3, we have $a_j' \in S^0(X \times \mathbb{R}^N)$ and $b_k' \in S^{-1}(X \times \mathbb{R}^N)$. It is straightforward that ${}^tLe^{i\varphi} = e^{i\varphi}$ and the formal adjoint $L = {}^t\big({}^tL\big)$ satisfies the conditions of the lemma. (More precisely, the coefficients of the operator L in (2.9) are given by $a_j := -a_j'$, $b_k := -b_k'$, and $c := c' - \sum_{j=1}^N \partial_{\theta_j} a_j - \sum_{k=1}^n \partial_{x_k} b_k$.) $\square$

Next we extend the oscillatory integrals (2.8) to the space of all symbols continuously:

Theorem 2.14. *Let φ be a phase function. Then, there exists a unique way to extend $I(a,\varphi)$ from $S^{-\infty}(X \times \mathbb{R}^N)$ to $S^\infty(X \times \mathbb{R}^N) := \cup_{m\in\mathbb{R}} S^m(X \times \mathbb{R}^N)$, such that*

$$S^m(X \times \mathbb{R}^N) \ni a \mapsto I(a,\varphi) \in \mathcal{D}'(X) \tag{2.11}$$

is continuous for all $m \in \mathbb{Z}$.[3]

Proof. Uniqueness is clear, as we have $S^{-\infty}(X \times \mathbb{R}^N)$ dense in $S^m(X \times \mathbb{R}^N)$ in any topology $S^{m'}(X \times \mathbb{R}^N)$ for $m' > m$ by Proposition 2.8. For existence, we integrate by parts formally. Namely, for $a \in S^{-\infty}(X \times \mathbb{R}^N)$ and $u \in C_0^\infty(X)$, we have:

$$\langle I(a, \varphi), u\rangle := \int_X \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x, \theta) u(x)\, d\theta\, dx.$$

Consider now the differential operator L given by Lemma 2.13. For $a \in S^m(X \times \mathbb{R}^N)$ and $k \in \mathbb{N}_0$, observe that $L^k\big(a(x, \theta)u(x)\big) \in S^{m-k}(X \times \mathbb{R}^N)$ by Proposition 2.6. Choose $k \in \mathbb{N}_0$ with $m - k < -N$ and define $I_k(a, \varphi)$ by the following expression:

$$\langle I_k(a, \varphi), u\rangle := \int_X \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} L^k\big(a(x, \theta)u(x)\big)\, d\theta\, dx. \tag{2.12}$$

Observe we have the following estimate, for any compact set $K \subset X$:

$$\sup_{K\times\mathbb{R}^N} \frac{|L^k(au)|}{(1 + |\theta|)^{m-k}} \leq f_{k,K}(a) \cdot \sum_{|\alpha|\leq k} \sup_K |\partial^\alpha u|, \tag{2.13}$$

where $f_{k,K}(a)$ is a suitable seminorm on $S^m(X\times\mathbb{R}^N)$ (involving the seminorms $P_{K,\alpha,\beta}$ defined in (2.5) for $|\alpha| + |\beta| \leq k$). Taking $K = \operatorname{supp}(u)$ this implies that

$$|\langle I_k(a, \varphi), u\rangle| \leq C f_{k,K}(a) \cdot \sum_{|\alpha|\leq k} \sup_K |\partial^\alpha u|,$$

which by Proposition 1.9 implies $u \in \mathcal{D}'(X)$ (in fact it is continuous as a map to $\mathcal{D}'^{(k)}(X)$). Moreover, it implies that $S^m(X\times\mathbb{R}^N) \ni a \mapsto I_k(a, \varphi) \in \mathcal{D}'(X)$ is continuous (by Proposition 1.7).

It is left to notice that by integrating by parts in (2.12), for any choice of k' with $m - k' < -N$, we have $I_{k'}(a, \varphi) = I_k(a, \varphi)$ and $I(a, \varphi) = I_k(a, \varphi)$ for any $a \in S^{-\infty}(X\times\mathbb{R}^N)$. Therefore, the extension is well-defined which completes the proof. □

Given a phase function φ on $X \times \big(\mathbb{R}^N \setminus 0\big)$, define the *critical set* of φ as

$$C_\varphi = \{(x, \theta) \in X \times \big(\mathbb{R}^N \setminus 0\big) \mid \partial_\theta \varphi(x, \theta) = 0\}. \tag{2.14}$$

Note that $\partial_\theta\varphi(x, \theta)$ is positively 0-homogeneous, so the set C_φ is closed (in $X\times(\mathbb{R}^N\setminus 0)$) and *conical*, that is, it is invariant under scaling by positive reals in the θ variable. The following statement shows that the singularities of $I(a, \varphi)$ are determined by the behaviour of a and φ near C_φ. Let $\Pi : X \times \mathbb{R}^N \to X$ be the projection to the first coordinate.

Proposition 2.15. *The following two claims hold for $a \in S^m(X \times \mathbb{R}^N)$:*

1. $\operatorname{sing\,supp} I(a, \varphi) \subset \Pi(C_\varphi)$;
2. *If a vanishes on a conical neighbourhood of C_φ, then $I(a, \varphi) \in C^\infty(X)$.*

Proof. For the first part, define $R_\varphi := X \setminus \Pi(C_\varphi)$, an open set. Note that the critical set of φ restricted to $R_\varphi \times (\mathbb{R}^N \setminus 0)$ is empty by definition, and that the restriction satisfies $I(a, \varphi)|_{R_\varphi} = I(a|_{R_\varphi\times\mathbb{R}^N}, \varphi|_{R_\varphi\times(\mathbb{R}^N\setminus0)})$. Therefore, Item 2 implies Item 1 (as in this case the critical set is empty).

[3]Moreover, if $k \in \mathbb{N}$ and $m - k < -N$, then $S^m \ni a \mapsto I(a, \varphi) \in \mathcal{D}'^{(k)}$ is continuous. Here, by $\mathcal{D}'^{(k)}(X)$ we denote the space of distributions of *order* $\leq k$, which is by definition the dual space of $C_0^k(X)$.

For the second part, we will follow the proof of Lemma 2.13. Introduce

$$^tL := \frac{1-\chi(\theta)}{i\widetilde{\Phi}} \cdot |\theta|^2 \sum_{j=1}^N \frac{\partial \bar{\varphi}}{\partial \theta_j}\frac{\partial}{\partial \theta_j} + \chi_0, \quad \widetilde{\Phi}(x,\theta) := |\theta|^2 \sum_{j=1}^N \Big|\frac{\partial \varphi}{\partial \theta_j}\Big|^2,$$

where $\chi, \chi_0 \in C^\infty(X \times \mathbb{R}^N)$ (where χ_0 is compactly supported in the θ variable), are chosen such that: χ is positively 0-homogeneous for large $|\theta|$, $\chi = 1$ on $C_\varphi \cup X \times \mathbb{B}(1)$, $\operatorname{supp}(\chi) \subset \{a = 0\} \cup X \times \mathbb{B}(2)$ where $\mathbb{B}(R) \subset \mathbb{R}^N$ denotes the ball of radius $R > 0$, and we set $\chi_0 = \chi_1\chi$, where $\chi_1 \in C_0^\infty(\mathbb{R}^N)$ satisfies $\chi_1 = 1$ on $\mathbb{B}(2)$. Notice in particular that these conditions imply $\operatorname{supp}(\chi_0 - \chi) \subset \{a = 0\}$. Then as in Lemma 2.13:

$$L = \sum_{j=1}^N a_j \frac{\partial}{\partial \theta_j} + c, \qquad a_j \in S^0(X \times \mathbb{R}^N), \quad c \in S^{-1}(X \times \mathbb{R}^N).$$

It is straightforward to compute $^tLe^{i\varphi} = (1 - \chi + \chi_0)e^{i\varphi}$. Therefore, if $a \in S^{-\infty}(X \times \mathbb{R}^N)$, we see that $I(a, \varphi) = I(L^k a, \varphi)$ for any $k \in \mathbb{N}_0$ (note that $(1 - \chi + \chi_0)a = a$ and L contains no x-derivatives). More generally if $a \in S^m(X \times \mathbb{R}^N)$, by Proposition 2.8, there is a sequence $a_j \in S^{-\infty}(X \times \mathbb{R}^N)$ (obtained by multiplying a by a cut-off function in θ), such that $a_j \to a$ in $S^{m'}(X \times \mathbb{R}^N)$ for $m' > m$; by Theorem 2.14 we have that:

$$I(a_i, \varphi) = I(L^k a_i, \varphi) \to I(L^k a, \varphi) = I(a, \varphi), \qquad i \to \infty,$$

where we use $L^k a_i \to L^k a$ in $S^{m-k}(X \times \mathbb{R}^N)$. Using $L^k a \in S^{m-k}(X \times \mathbb{R}^N)$, and taking k to be arbitrarily large, we deduce that $I(a, \varphi) \in C^\infty(X)$, completing the proof. □

As an example of previous theory, consider $X \times Y \times \mathbb{R}^N$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are open sets. An operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is called a *Fourier Integral Operator* (in short *FIO*), if there exists an $a \in S^m(X \times Y \times \mathbb{R}^N)$ and a phase function φ such that

$$Au(x) = \int_Y \int_{\mathbb{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) u(y)\, d\theta\, dy, \qquad u \in C_0^\infty(Y). \tag{2.15}$$

Here we view the oscillatory integral $K_A = I(a, \varphi) \in \mathcal{D}'(X \times Y)$ as a Schwartz kernel generating an operator A via (recall Theorem 1.16)

$$\langle Au, v\rangle := \langle K_A, v \otimes u\rangle, \qquad v \in C_0^\infty(X).$$

In the particular case when $X = Y$, $N = n$, $\varphi(x, y, \xi) = (x - y) \cdot \xi$, and $K_A = (2\pi)^{-n} I(a, \varphi)$ (the extra $(2\pi)^{-n}$ factor is to align this definition with (2.1)), the operator A is called a *pseudodifferential operator* (in short *PDO*). More explicitly, for PDOs we have

$$\operatorname{Op}(a)u(x) := Au(x) = (2\pi)^{-n} \int_X \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y)\, d\xi\, dy, \qquad u \in C_0^\infty(X), \tag{2.16}$$

where $\operatorname{Op}(\bullet)$ is referred to as *quantisation of a*. Denote the space of such PDOs as $\Psi^m(X)$ and say that $A \in \Psi^m(X)$ is of order m. Write $\Psi^{-\infty}(X)$ for the set of operators of order $-\infty$, that is, $\Psi^{-\infty}(X) := \cap_{m\in\mathbb{Z}} \Psi^m(X)$, and call elements of $\Psi^{-\infty}(X)$ *smoothing*.

Next, we show some basic properties of PDOs. Denote by $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ the diagonal.

Proposition 2.16. *Let $A \in \Psi^m(X)$ for some $m \in \mathbb{R}$ or $m = -\infty$. Then:*

1. *A is a continuous map $A : C_0^\infty(X) \to C^\infty(X)$;*
2. *There exists a well-defined transpose operator $^tA \in \Psi^m(X)$ satisfying $\langle {}^tAv, u\rangle = \langle Au, v\rangle$ for all $u, v \in C_0^\infty(X)$;*

3. *A has a unique continuous extension $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$ (recall $\mathcal{E}'(X)$ was introduced in Proposition 1.14 as the set of distributions with compact support);*
4. *K_A is smooth outside of Δ, i.e. $K_A|_{X\times X\setminus\Delta} \in C^\infty(X \times X \setminus \Delta)$;*
5. *For all $u \in C_0^\infty(X)$, we have $\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)$, that is, A is a* pseudolocal *operator.*

Proof. Item 1. Note that $d_{y,\xi}\varphi = (-\xi, x - y) \neq 0$ for all points $(x, y, \xi) \in X \times X \times (\mathbb{R}^n \setminus 0)$ (i.e. $\varphi(x, \bullet, \bullet)$ is a phase function on $X \times (\mathbb{R}^N \setminus 0)$ for every fixed $x \in X$). By the proof of Lemma 2.13, there is an $L = \sum_{j=1}^n a_j\partial_{\xi_j} + \sum_{k=1}^n b_k\partial_{y_k} + c$, where $a_j \in S^0(X \times X \times \mathbb{R}^n)$, $b_k, c \in S^{-1}(X \times X \times \mathbb{R}^n)$, such that ${}^tL(e^{i\varphi}) = e^{i\varphi}$. More explicitly, as in Lemma 2.13 we have

$$ {}^tL = \frac{1-\chi(\xi)}{i\Phi(x,y,\xi)}\Big(|\xi|^2\sum_{j=1}^n (x_j - y_j)\partial_{\xi_j} - \sum_{k=1}^n \xi_k\partial_{y_k}\Big) + \chi(\xi), \quad \Phi(x,y,\xi) = |\xi|^2(|x-y|^2+1). $$

Then for $u, v \in C_0^\infty(X)$ and $k \in \mathbb{N}_0$ with $m - k < -n - k$, by definition (see Theorem 2.14):

$$ \langle Au, v\rangle = \langle K_A, v \otimes u\rangle = \int_X\int_X\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} v(x) L^k\big(a(x,y,\xi)u(y)\big)\, d\xi\, dy\, dx, $$

where we used that L has no y-derivatives. Therefore, as $L^k(au) \in S^{m-k}(X \times X \times \mathbb{R}^N)$, Au is the C^k-regular function given by the oscillatory integral

$$ Au(x) = (2\pi)^{-n}\int_X\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} L^k\big(a(x,y,\xi)u(y)\big)\, d\xi\, dy, \tag{2.17} $$

and since k can be taken arbitrarily large, we conclude $Au \in C^\infty(X)$. The continuity of $A : C_0^\infty(X) \to C^\infty(X)$ follows from this formula and the estimate in (2.13). (More precisely, by Proposition 1.7 it suffices to show for any compact set $K \subset X$ and any multi-index α, that $\sup_{x\in K}|\partial^\alpha(Au)(x)|$ is a bounded seminorm on $C_0^\infty(X)$. By Proposition 1.9 (the analogous version for seminorms), it suffices to show an estimate of the form (1.2), which follows from (2.13) and its variant for $\partial_x^\alpha L(au)$ (the estimate (2.13) works for $\alpha = 0$)).

---

end of Lecture 4, 11.10.2022

Item 2. For the second part, in view of Schwartz kernel Theorem 1.16, the transpose of A, ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$, is given by exchanging the roles of x and y:

$$ \langle {}^tAv, u\rangle := \langle Au, v\rangle = \langle K_A, v \otimes u\rangle =: \langle K_{{}^tA}, u \otimes v\rangle, \qquad u, v \in C_0^\infty(X), $$

where $K_{{}^tA}(x,y) = K_A(y,x)$ is the Schwartz kernel of ${}^tA$. It follows that $K_{{}^tA} = (2\pi)^{-n}I(\widetilde{a}, (x-y)\cdot\xi)$, where $\widetilde{a}(x,y,\xi) = a(y,x,-\xi) \in S^m(X \times X \times \mathbb{R}^n)$, so ${}^tA \in \Psi^m(X)$.

Item 3. By Items 1 and 2, we get ${}^tA : C_0^\infty(X) \to C^\infty(X)$. Therefore, there is a unique continuous extension given by (see also (1.7))

$$ \langle Au, v\rangle := \langle u, {}^tAv\rangle, \qquad u \in \mathcal{E}'(X), \quad v \in C_0^\infty(X). $$

Item 4. Since the critical set of $\varphi = (x-y)\cdot\xi$ satisfies $C_\varphi = \Delta \times \big(\mathbb{R}^n \setminus 0\big)$, the third point follows from Proposition 2.15, first item.

Item 5. Let $u \in \mathcal{E}'(X)$ and take $x_0 \in X \setminus \operatorname{sing\,supp}(u)$. Take $\psi_1, \psi_2 \in C_0^\infty(X)$ with $\operatorname{supp}(\psi_1) \cap \operatorname{supp}(\psi_2) = \emptyset$, $\psi_1 = 1$ near x_0 and $\psi_2 = 1$ near $\operatorname{sing\,supp}(u)$. Let us write

$$ \psi_1 Au = \psi_1 A\big((1-\psi_2)u\big) + \psi_1 A(\psi_2 u). \tag{2.18} $$

It suffices to show that the two terms on the right hand side are in $C^\infty(X)$. In fact, by Item 1 above we have $A\big((1-\psi_2)u\big) \in C^\infty(X)$. Next, since by Item 4 above $\psi_1(x)K_A(x,y)\psi_2(y) \in$

$C^\infty(X \times X)$, which is the Schwartz kernel of the operator $u \mapsto \psi_1 A(\psi_2 u)$, we get that the second term is also in $C^\infty(X)$ by Proposition 2.17 below.

□

Finally we show that smoothing operators arise as operators that 'smoothen out' every distribution. (The equivalence of first two points is 'standard' in distribution theory.)

Proposition 2.17. *For a continuous operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ with Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$ (see Theorem 1.16), the following three statements are equivalent:*

1. *A extends to a continuous operator $A : \mathcal{E}'(Y) \to C^\infty(X)$;*
2. *$K_A \in C^\infty(X \times Y)$;*
3. *(when $X = Y$) A is smoothing, that is, $A \in \Psi^{-\infty}(X)$.*

Proof. 2. $\implies$ 1. Given $u \in \mathcal{E}'(Y)$, we may define $Au(x) := \varphi(x) := \langle u, K_A(x, \bullet)\rangle$, where $K_A(x, \bullet) \in C^\infty(Y)$ is considered as a smooth function, and we claim that φ is in fact a smooth function. Firstly, if $x_i \to x$, then $\varphi(x_i) \to \varphi(x)$ since $K_A(x_i, \bullet) \to K_A(x, \bullet)$ in $C^\infty(Y)$ and by continuity of u, showing $\varphi \in C^0(X)$. Next, for $i \in \{1, \dots, n\}$ introduce

$$a_i(x) := \langle u, \partial_{x_i} K(x, \bullet)\rangle,$$

and by the analogous argument a_i is continuous. We claim that $\partial_{x_i}\varphi = a_i$. To see this, for any $x \in X$, and $\varepsilon > 0$ small enough, we have:

$$\begin{aligned}\int_0^\varepsilon a_i(x + s\mathbf{e}_i)ds &= \int_0^\varepsilon \langle u, \partial_{x_i} K(x + s\mathbf{e}_i, \bullet)\rangle ds\\ &= \Big\langle u, \int_0^\varepsilon \partial_{x_i} K(x + s\mathbf{e}_i, \bullet)ds\Big\rangle = \varphi(x + \varepsilon\mathbf{e}_i) - \varphi(x),\end{aligned}$$

where in passing to the second line, we wrote the $\int_0^\varepsilon$ integral as a limit of Riemann sums, and used that they converge uniformly and eventually in $C^\infty(Y)$. Taking the limit $\varepsilon \to 0$ and using the Fundamental Theorem of Calculus proves the claim, and so $\varphi \in C^1(X)$. Iterating this argument, we get $\varphi \in C^\infty(X)$, proving the initial claim. The continuity of the map A follows similarly.

1. $\implies$ 2. Let us 'plug in' suitable distributions. Define $K(x, y) := \langle A\delta_y, \delta_x\rangle$. We claim that $K \in C^\infty(X \times Y)$. To check continuity, notice that if $(x_i, y_i) \to (x, y)$, then $A\delta_{y_i} \to A\delta_y$ in $C^\infty(X)$, and so $K(x_i, y_i) = \langle A\delta_{y_i}, \delta_{x_i}\rangle \to K(x, y)$. For differentiability, notice that for any (x, y), any $i \in \{1, \dots, n\}$, and $|s|$ small enough, we have

$$\frac{K(x + s\mathbf{e}_i, y) - K(x, y)}{s} = \Big\langle A\delta_y, \frac{\delta_{x+s\mathbf{e}_i} - \delta_x}{s}\Big\rangle \to \langle A\delta_y, \partial_{x_i}\delta_x\rangle, \quad s \to 0,$$

since $\frac{\delta_{x+s\mathbf{e}_i} - \delta_x}{s} \to \partial_{x_i}\delta_x$ in $\mathcal{E}'(X)$. By the analogous argument as in the previous paragraph, $\langle A\delta_y, \partial_{x_i}\delta_x\rangle \in C^0(X \times Y)$, and we conclude that $K \in C^1(X \times Y)$. Iterating this argument, we get that K is smooth.

It suffices to prove $K = K_A$. Let B be the operator with the Schwartz kernel K. Notice that for any $y \in Y$, $A\delta_y$ is a smooth function which by definition agrees with $B\delta_y$ at any $x \in X$, hence $A\delta_y = B\delta_y$. It now suffices to show that the space spanned by Dirac distributions is dense in the topology of distributions. That is, since $C^\infty(Y) \subset \mathcal{D}'(Y)$ is dense (see Example 2.18 below), it suffices to show that any $\varphi \in C^\infty(Y)$ is a limit in $\mathcal{D}'(Y)$ of a linear combination of Dirac masses. To do this, for each $k \in \mathbb{N}$, enumerate $Y \cap \left(\frac{\mathbb{Z}}{k}\right)^n$ by $(y_{k,\alpha})_\alpha$. Define $\varphi_k := \sum_\alpha \frac{\varphi(y_{k,\alpha})}{k^n} \cdot \delta_{y_{k,\alpha}} \in \mathcal{D}'(Y)$ and notice that for any $\psi \in C_0^\infty(Y)$ we have:

$$\int_Y \varphi_k(y) \cdot \psi(y)\ dy = \sum_\alpha \frac{\varphi(y_{k,\alpha}) \cdot \psi(y_{k,\alpha})}{k^n} \to_{k\to\infty} \int_Y \varphi(y) \cdot \psi(y)\ dy,$$

where in the last limit we used the convergence of Riemann sums. It follows that $\varphi_k \to \varphi$ in $\mathcal{D}'(Y)$ as $k \to \infty$. Consequently, by continuity $A\psi = B\psi$, so $A \equiv B$ which implies $K = K_A$ completing the proof (by Theorem 1.16, operators uniquely determine their Schwartz kernels).

2. $\iff$ 3. when $X = Y$. Assume first $A \in \Psi^{-\infty}(X)$. By definition, the symbol a defining A satisfies $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$ so the oscillatory integral K_A defined in (2.16) is smooth, showing Item 2.

For the converse, assume we have a smooth Schwartz kernel $K_A \in C^\infty(X \times X)$. In particular we may choose $a(x,y,\xi) := (2\pi)^n K_A(x,y)e^{-i(x-y)\cdot\xi}\chi(\xi) \in S^{-\infty}(X \times X \times \mathbb{R}^n)$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\int \chi(\chi)d\xi = 1$. It follows that $K_A = (2\pi)^{-n} I(a, (x-y)\cdot\xi)$ and so $A \in \Psi^{-\infty}(X)$, i.e. it satisfies (2.16). □

Example 2.18 (Convolution operators are smoothing)**.** Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ that integrates to 1, i.e. $\int_{\mathbb{R}^n} \chi(x)\ dx = 1$ and $\chi = 1$ in a neighbourhood of zero. For $\varepsilon > 0$, set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi\left(\frac{x}{\varepsilon}\right)$ and define $A_\varepsilon u := \chi_\varepsilon * u$ for $u \in C_0^\infty(\mathbb{R}^n)$ (recall $*$ denotes convolution, see Proposition 1.19, Item 3). By definition of convolution, the Schwartz kernel K_ε of A_ε is equal to $K_\varepsilon(x,y) = \chi_\varepsilon(x-y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. So by Proposition 2.17, $A_\varepsilon u \in C^\infty(\mathbb{R}^n)$ for any $u \in \mathcal{E}'(\mathbb{R}^n)$. It can be shown that $A_\varepsilon u \to u$ as $\varepsilon \to 0$ in the distribution sense (exercise), so smooth functions are dense in the space of distributions.

3. Method of stationary phase

Aim. In this section we consider integrals of the form, for $\varphi \in C^\infty(X;\mathbb{R})$ and $u \in C_0^\infty(X)$:

$$I(\lambda) := \int_X e^{i\lambda\varphi(x)} u(x)\ dx, \qquad \lambda > 0. \tag{3.1}$$

We are interested in the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$.

As a first result, we record:

Lemma 3.1 (Non-stationary phase lemma)**.** *If $d\varphi \neq 0$ on X, then for all $N > 0$ there exists $C = C(N, \varphi, \operatorname{supp}(u)) > 0$ such that:*

$$|I(\lambda)| \leq C\lambda^{-N} \times \Big(\sum_{|\alpha|\leq N} \sup_x |\partial^\alpha u(x)|\Big).$$

Proof. Let ${}^tL := \frac{1}{i\lambda|d\varphi|^2}\sum_{j=1}^n \partial_{x_j}\varphi \cdot \partial_{x_j}$, similarly as in Lemma 2.13, so that ${}^tLe^{i\lambda\varphi} = e^{i\lambda\varphi}$. Integrating by parts k times, we get

$$I(\lambda) = \int_X e^{i\lambda\varphi} L^k u\ dx,$$

and after observing that $|L^k u|$ is bounded by λ^{-k} times a constant depending on φ, $\operatorname{supp}(u)$, and k derivatives of u, the result follows. □

Next, we turn to the more difficult case when φ has critical points. We will see that $I(\lambda)$ has a polynomial expansion in inverse powers of λ near critical points of φ with coefficients depending on u.

Recall that $x_0 \in X$ is a *non-degenerate* critical point of φ if $d\varphi(x_0) = 0$ and $\det \varphi''(x_0) \neq 0$. Here the symmetric matrix $\varphi''(x) = \big(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\big)_{i,j=1}^n (x)$ denotes the *Hessian*.

Lemma 3.2 (Morse lemma)**.** *If x_0 is a non-degenerate critical point of φ, there exists a local diffeomorphism $\mathcal{H} : U \to V \subset \mathbb{R}^n$ with $x_0 \in U$, $\mathcal{H}(x_0) = 0$, such that:*

$$\varphi \circ \mathcal{H}^{-1}(x_1, \dots, x_n) = \varphi(x_0) + \frac{1}{2}(x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_n^2),$$

where r is the maximal dimension of a subspace $V \subset \mathbb{R}^n$ on which the matrix $\varphi''(x_0)$ is positive definite.

Proof. For instance, see [4, Lemma 2.1]. $\square$

If r is the value from Lemma 3.2, define the *signature* of $\varphi''(x_0)$ by $\operatorname{sgn} \varphi''(x_0) = 2r - n$. Morse lemma reduces the study of (3.1) to considering a symmetric, non-degenerate matrix Q with signature $\operatorname{sgn} Q = 2r - n$. Arguing as in the derivation of (1.11), it is possible to derive the following identity (the Fourier transform of a 'complex Gaussian'):

$$\mathcal{F}\big(e^{\frac{i}{2}\langle x, Qx \rangle}\big)(\xi) = (2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4} \operatorname{sgn} Q} |\det Q|^{-\frac{1}{2}} e^{-\frac{i}{2}\langle \xi, Q^{-1}\xi \rangle}, \tag{3.2}$$

where we make use of the following one dimensional identity (analogous to (1.12)):

$$\mathcal{F}(e^{\pm \frac{i}{2}x^2})(\xi) = \sqrt{2\pi} \cdot e^{\pm \frac{i\pi}{4}} e^{\pm \frac{i}{2}\xi^2}.$$

Note that the Fourier transforms are interpreted as Fourier transforms of Schwartz distributions (see Proposition 1.19, Item 6).

Theorem 3.3 (Stationary phase lemma)**.** *Let $\varphi \in C^\infty(X; \mathbb{R})$ such that $x_0 \in X$ is a non-degenerate critical point of φ in X, and $d\varphi(x) \neq 0$ for $x \neq x_0$. For $\nu \in \mathbb{N}_0$ there exist differential operators $A_{2\nu}$ of order at most 2ν (depending only on φ), such that for every compact set $K \subset X$ and $N \in \mathbb{N}_0$, there exists $C = C_{K,N} > 0$, such that for all $u \in C_0^\infty(K)$:*

$$\begin{aligned}&\Big| \int_X e^{i\lambda\varphi(x)} u(x)\, dx - e^{i\lambda\varphi(x_0)} \Big(\sum_{\nu=0}^{N-1} (A_{2\nu}(x, \partial_x)u)(x_0) \cdot \lambda^{-\nu - \frac{n}{2}} \Big) \Big| \\ &\leq C\lambda^{-N-\frac{n}{2}} \sum_{|\alpha| \leq 2N+n+1} \sup_{x \in K} |\partial^\alpha u(x)|, \qquad \lambda \geq 1.\end{aligned}$$

Moreover, we have $A_0 = \frac{(2\pi)^{\frac{n}{2}} \cdot e^{i\frac{\pi}{4} \operatorname{sgn} \varphi''(x_0)}}{|\det \varphi''(x_0)|^{\frac{1}{2}}}$.

Proof. Step 1: reduction to quadratic phase function. Let $\chi \in C_0^\infty(X)$ such that $\chi = 1$ near x_0 and $\operatorname{supp}(\chi) \subset U$, where U is the open neighbourhood of x_0 given by Lemma 3.2. Then

$$\int_X e^{i\lambda\varphi(x)} u(x)\, dx = \int_X e^{i\lambda\varphi}(1 - \chi)u\, dx + e^{i\lambda\varphi(x_0)} \int_V e^{\frac{i}{2}\lambda\langle y, Qy \rangle} v(y)\, dy,$$

where $v(y) = (\chi \cdot u)(\mathcal{H}^{-1}y) \cdot \det J(y)$, $J_{ij}(y) := \frac{\partial \mathcal{H}_i^{-1}}{\partial y_j}$ is the Jacobian matrix of $\mathcal{H}^{-1} = (\mathcal{H}_1^{-1}, \dots, \mathcal{H}_n^{-1})$, and $Q = \begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & -\mathrm{Id}_{n-r} \end{pmatrix}$. By Lemma 3.1, the first term is decaying faster than any polynomial in λ, i.e. for all $N > 0$ there is a $C_N > 0$ that bounds the term by $C_N\lambda^{-N}$. For the second term, it suffices to show the claim for the special case of a quadratic function (note that $J(y)$ depends on φ). Finally, to compute A_0, we will simply use (3.6) below together with the fact that $Q = J(0)^T\varphi''(x_0)J(0)$ which implies $(\det J(0))^2 \cdot \det\varphi''(x_0) = \det Q$. (For this, apply the chain rule to compute for $\psi := \varphi\circ\mathcal{H}^{-1}$ that $\frac{\partial^2\psi}{\partial y_i\partial y_j} = \sum_{k,\ell}\varphi''_{k\ell}(x_0)\frac{\partial\mathcal{H}_\ell^{-1}}{\partial y_j}\frac{\partial\mathcal{H}_k^{-1}}{\partial y_i}$ at $y = 0$.)

Step 2: quadratic phase function. By Parseval's identity, Proposition 1.19 7. (i.e. that $(2\pi)^n\langle f, g\rangle = \langle\widehat{f},\widehat{g}\rangle$), we obtain:

$$\int_X e^{\frac{i}{2}\lambda\langle x,Qx\rangle}u(x)\;dx = (2\pi)^{-\frac{n}{2}}e^{i\frac{\pi}{4}\operatorname{sgn}Q}|\det Q|^{-\frac{1}{2}}\lambda^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{i}{2\lambda}\langle\xi,Q^{-1}\xi\rangle}\widehat{u}(\xi)\;d\xi. \tag{3.3}$$

By Taylor's formula we have that for all $N\in\mathbb{N}_0$:

$$\Big|e^{it} - \sum_{k=0}^{N-1}\frac{(it)^k}{k!}\Big| \le \frac{|t|^N}{N!}, \qquad t\in\mathbb{R}.$$

Therefore, we obtain for some $R_N(\xi,\lambda)$:

$$e^{-\frac{i}{2\lambda}\langle\xi,Q^{-1}\xi\rangle} = \sum_{k=0}^{N-1}\frac{1}{k!}\Big(\frac{1}{2\lambda i}\langle\xi,Q^{-1}\xi\rangle\Big)^k + R_N(\xi,\lambda), \quad |R_N(\xi,\lambda)| \le \frac{1}{(2\lambda)^N N!}|\langle\xi,Q^{-1}\xi\rangle|^N. \tag{3.4}$$

Next, by the properties of the Fourier transform (see Proposition 1.19, Items 5 and 6), we obtain

$$\int_{\mathbb{R}^n}\langle\xi,Q^{-1}\xi\rangle^k\widehat{u}(\xi)\;d\xi = (2\pi)^n\mathcal{F}^{-1}\big(\langle\xi,Q^{-1}\xi\rangle^k\widehat{u}(\xi)\big)(0) = (2\pi)^n\big(\langle D_x,Q^{-1}D_x\rangle^k u\big)(0). \tag{3.5}$$

Here $\langle D_x,Q^{-1}D_x\rangle$ denotes the constant coefficient differential operator $-Q^{-1}_{ij}\partial_{x_i}\partial_{x_j}$ associated to Q^{-1}. Combining (3.3), (3.4), and (3.5), we obtain:

$$\int_{\mathbb{R}^n}e^{\frac{i}{2}\lambda\langle x,Qx\rangle}u(x)\;dx = \sum_{k=0}^{N-1}\frac{(2\pi)^{\frac{n}{2}}e^{i\frac{\pi}{4}\operatorname{sgn}Q}}{k!|\det Q|^{\frac{1}{2}}\lambda^{k+\frac{n}{2}}}\Big(\frac{1}{2i}\langle D_x,Q^{-1}D_x\rangle\Big)^k u(0) + S_N(u,\lambda), \tag{3.6}$$

where the remainder term $S_N(u,\lambda)$ can be estimated as, using (3.4)

$$\begin{aligned}|S_N(u,\lambda)| &\le |\det Q|^{-\frac{1}{2}}\lambda^{-\frac{n}{2}}(2\lambda)^{-N}(N!)^{-1}\int_{\mathbb{R}^n}|\langle\xi,Q^{-1}\xi\rangle|^N|\widehat{u}(\xi)|\;d\xi\\ &\le \frac{C}{\lambda^{\frac{n}{2}+N}N!}\times\Big(\sum_{|\alpha|\le 2N+n+1}\sup_x|D^\alpha u(x)|\Big),\end{aligned} \tag{3.7}$$

where $C = C(\operatorname{supp}(u),N,Q) > 0$, and in the second line we used Proposition 1.19, Items 1 and 5, to estimate the integral. This completes the proof. $\square$

Let us now specialise to $Q = -\begin{pmatrix}0 & \mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix}$, where each block is an $n\times n$ matrix, and switch to $(x,y)\in\mathbb{R}^{2n}$ variables, and record (3.6) in this case for future use. Then

$$\frac{1}{2}\langle(x,y),Q(x,y)\rangle = -x\cdot y, \qquad \langle D_{(x,y)},Q^{-1}D_{(x,y)}\rangle = 2\sum_j\partial_{x_j}\partial_{y_j}.$$

Note also that $\operatorname{sgn}(Q) = 0$, $Q^{-1} = Q$, and $|\det Q| = 1$. By equation (3.6)

$$\begin{aligned}\left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\lambda x\cdot y} u(x,y)\, dx\, dy &= \sum_{k=0}^{N-1} \frac{1}{k!\lambda^k}\Big(\frac{1}{i}\sum_{j=1}^n \partial_{x_j}\partial_{y_j}\Big)^k u(0,0) + S_N(u,\lambda)\\ &= \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!\lambda^{|\alpha|} i^{|\alpha|}} \partial_x^\alpha \partial_y^\alpha u(0,0) + S_N(u,\lambda),\end{aligned} \tag{3.8}$$

where $S_N(\lambda)$ satisfies the estimate from Theorem 3.3.

end of Lecture 5, 18.10.2022

Remark 3.4. This method is useful in various areas of mathematics and physics, for instance: in PDEs, Chern-Simons theory (infinite dimensional setting), proving various formulas.

4. Algebra of pseudodifferential operators

Aim. In this section we show basic properties of pseudodifferential operators. More precisely we show that, up to smoothing operators, the symbols of PDOs can always be taken to be independent of the y-variable, i.e. in *left quantisation*. Moreover, we show that composition of PDOs is still a PDO, and invariance under coordinate changes.

4.1. Compositions.

Say a closet set $\mathcal{C} \subset X \times Y$ is *proper* if the projections $\Pi_x : X \times Y \to X$, $(x,y) \mapsto x$, and $\Pi_y : X \times Y \to Y$, $(x,y) \mapsto y$, are proper maps when restricted to $\mathcal{C}$. Here we recall a map $f : V \to W$ between topological spaces is proper if $f^{-1}(K) \subset V$ is compact for all compact sets $K \subset W$. An operator $A : C_0^\infty(Y) \to C^\infty(X)$ is *properly supported* if $\operatorname{supp}(K_A) \subset X \times Y$ is proper.

Proposition 4.1. *Assume $A \in \Psi^m(X)$ is properly supported. Then for all $u \in C_0^\infty(X)$:*

$$\operatorname{supp}(Au) \subset \Pi_x\big(\operatorname{supp} K_A \cap X \times \operatorname{supp}(u)\big) = \Pi_x\big(\operatorname{supp} K_A \cap \Pi_y^{-1} \operatorname{supp}(u)\big). \tag{4.1}$$

Moreover, A extends as a continuous map:

$$C_0^\infty(X) \to C_0^\infty(X), \quad C^\infty(X) \to C^\infty(X), \quad \mathcal{E}'(X) \to \mathcal{E}'(X), \quad \mathcal{D}'(X) \to \mathcal{D}'(X).$$

Proof. To show (4.1), simply observe that if $v \in C_0^\infty(X)$ is such that $\operatorname{supp}(v)$ is contained in the complement of $\Pi_x\big(\operatorname{supp} K_A \cap \Pi_y^{-1} \operatorname{supp}(u)\big)$, then $\operatorname{supp}(K_A) \cap \operatorname{supp}(v) \times \operatorname{supp}(u) = \emptyset$, so $\langle Au, v\rangle = \langle K_A, v \otimes u\rangle = 0$.

Next, (4.1) shows that $Au \in C_0^\infty(X)$, so $A : C_0^\infty(X) \to C_0^\infty(X)$ is a continuous mapping. Here continuity follows by definition of topology on $C_0^\infty(X)$ and the continuity of $A : C_0^\infty(X) \to C^\infty(X)$ shown in Proposition 2.16, Item 1. Similarly A extends to a map $A : C^\infty(X) \to C^\infty(X)$. Analogous arguments apply to ${}^tA \in \Psi^m(X)$ (see Proposition 2.16, Item 2), and so by duality A extends to a map $A : \mathcal{D}'(X) \to \mathcal{D}'(X)$ and $A : \mathcal{E}'(X) \to \mathcal{E}'(X)$. $\square$

Any pseudodifferential operator can be decomposed into smoothing and properly supported factors.

Proposition 4.2. *Any $A \in \Psi^m(X)$ can be written as $A = A_0 + A_1$, where $A_1 \in \Psi^{-\infty}(X)$ is smoothing and $A_0 \in \Psi^m(X)$ is properly supported.*

Proof. Let $\chi \in C^\infty(X \times X)$ be such that $\chi = 1$ in a neighbourhood of the diagonal $\Delta(X) \subset X \times X$ and $\operatorname{supp}\chi \subset X \times X$ proper. Indeed, such χ exists since for each $x \in X$, there exist open neighbourhoods $V_x \subset U_x$ of x, contained in a ball of radius 1 around x, such that the closures satisfy $\overline{U}_x \subset X$ (i.e. U_x is a positive distance from the boundary of X) and $\overline{V}_x \subset U_x$. Let $\mathcal{U} := \cup_{x\in X} U_x \times U_x$ and $\mathcal{V} := \cup_{x \in X} V_x \times V_x$. Then $\overline{\mathcal{V}} \subset \mathcal{U}$ by construction and the existence of χ follows from the existence of a partition of unity with respect to the open cover $\{\mathcal{U}, X \times X \setminus \overline{\mathcal{V}}\}$ of $X \times X$.

Next, by definition $A = \mathrm{Op}(a)$ for some $a \in S^m(X \times X \times \mathbb{R}^n)$. Write:

$$A = A_0 + A_1, \quad A_0 := \mathrm{Op}(\chi \cdot a) \in \Psi^m(X), \quad A_1 := \mathrm{Op}\big((1-\chi)\cdot a\big).$$

Then the Schwartz kernel $K_{A_1} = (1-\chi)\cdot K_A \in C^\infty(X \times X)$ by Proposition 2.16, Item 4, so by Proposition 2.17 we have $A_1 \in \Psi^{-\infty}(X)$, and $K_{A_0} = \chi \cdot K_A$ is properly supported by the defining properties of χ, completing the proof. $\square$

The next result says that up to smoothing, it only suffices to consider left quantisation, i.e. symbols depending only on (x, ξ).

Theorem 4.3. *Let $a \in S^m(X \times X \times \mathbb{R}^n)$ and assume $A := \mathrm{Op}(a) \in \Psi^m(X)$ is properly supported. Then*

$$b(x,\xi) := e^{-ix\cdot\xi} A(e^{i\bullet\cdot\xi}) \tag{4.2}$$

satisfies $b \in S^m(X \times \mathbb{R}^n)$ and has an asymptotic development (recall (2.7))

$$b(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!}\big(\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\big)\big|_{y=x}. \tag{4.3}$$

Moreover, $Au(x) = \mathrm{Op}(b)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} b(x,\xi)\widehat{u}(\xi)\, d\xi$ for $u \in C_0^\infty(X)$.

Call $\sigma_A(x,\xi) := b(x,\xi)$ the *complete symbol of A*.

Proof. Step 0: motivation. For $y \in \mathbb{R}^n$, the Fourier transform (operating on $\mathcal{S}'(\mathbb{R}^n)$) of the Dirac delta distribution δ_y is given by $\mathcal{F}\delta_y(\xi) = e^{-iy\cdot\xi}$, and so $\delta_y(x) = (2\pi)^{-n}\mathcal{F}(e^{i\bullet\cdot y})(x)$. We obtain that for any $c \in S^m(X \times \mathbb{R}^n)$

$$\begin{aligned} e^{ix\cdot\xi}\big(\mathrm{Op}(c)e^{i\bullet\cdot\xi}\big)(x) &= e^{-ix\cdot\xi}\mathcal{F}^{-1}\big(c(x,\bullet)\mathcal{F}e^{i\bullet\cdot\xi}\big)(x) = e^{-ix\cdot\xi}(2\pi)^n\mathcal{F}^{-1}\big(c(x,\bullet)\delta_\xi(\bullet)\big)(x)\\ &= e^{-ix\cdot\xi}c(x,\xi)(2\pi)^n\mathcal{F}^{-1}\big(\delta_\xi(\bullet)\big)(x) = c(x,\xi). \end{aligned}$$

Therefore, if b satisfies $\mathrm{Op}(b) = A$ and does not depend on the y-variable, then (4.2) necessarily 'reconstructs' b from A. In what follows we show b satisfies the required properties.

Step 1. By multiplying $a(x,y,\xi)$ by a cut-off $\chi \in C^\infty(X \times X)$ with $\chi = 1$ near $\operatorname{supp} K_A \cup \Delta(X)$ and with $\operatorname{supp}\chi \subset X \times X$ proper, we may assume that $\Pi_{(x,y)}\operatorname{supp}(a) = \{(x,y) \in X \times X \mid \exists \xi \in \mathbb{R}^n,\ a(x,y,\xi) \neq 0\}$ is proper, where χ is constructed similarly as in Proposition 4.2. Then $\mathrm{Op}(\chi\cdot a) = \mathrm{Op}(a) = A$ (as $\chi \cdot K_A = K_A$) and since $\chi = 1$ near $\Delta(X)$, the asymptotic expansion in (4.2) is not affected. Write

$$e^{-ix\cdot\xi}\big(Ae^{i\bullet\cdot\xi}\big)(x) = (2\pi)^{-n}\int_X\int_{\mathbb{R}^n} a(x,y,\theta)e^{i(x-y)\cdot(\theta-\xi)}\, d\theta\, dy =: b(x,\xi). \tag{4.4}$$

Note that $b \in C^\infty(X \times \mathbb{R}^n)$ by the proof of Proposition 2.16, Item 1 (more precisely, by the integration by parts formula in (2.17) for $u(y) = e^{iy\cdot\xi}$). We next evaluate the integral (4.4) using the method of stationary phase and the phase function $\varphi(y,\theta) = (x-y)\cdot(\theta-\xi)$ with the only critical point at $(y,\theta)=(x,\xi)$. Since the property $b\in S^m(X\times\mathbb{R}^n)$ depends only on large $|\xi|$, we can restrict to $|\xi|\geq 1$.

Step 2: reduction to a compact set in θ. Fix a cut-off function $\chi\in C_0^\infty([0,\infty);[0,1])$ such that $\chi=1$ on $[0,\frac{1}{3}]$ and $\operatorname{supp}\chi\subset[0,\frac{1}{2})$. For $|\xi|\geq 1$ and x in a compact set $K\subset X$, set:

$$b_2(x,\xi):=(2\pi)^{-n}\int_X\int_{\mathbb{R}^n}a(x,y,\theta)\Big(1-\chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)\Big)e^{i(x-y)\cdot(\theta-\xi)}\,d\theta\,dy. \tag{4.5}$$

Note that for x in a compact set of X, the integral in the y-variable may be taken to be over a compact set (since $\Pi_{(x,y)}\operatorname{supp}(a)$ is proper by Step 1). Let us integrate by parts with $L:=\frac{1}{|\theta-\xi|^2}\sum_{k=1}^n(\xi_k-\theta_k)D_{y_k}$. Observe that ${}^tL=-L$ and $Le^{i(x-y)\cdot(\theta-\xi)}=e^{i(x-y)\cdot(\theta-\xi)}$. Observe also that on the support of $1-\chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)$, we have $|\theta-\xi|\geq\frac{1}{3}|\xi|$ so by triangle inequality

$$1+|\theta|+|\xi|\geq|\theta-\xi|\geq\frac{1}{10}(1+|\theta|+|\xi|). \tag{4.6}$$

Then (where $a(1-\chi)$ below stands for the function $a(x,y,\theta)\Big(1-\chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)\Big)$):

$$\begin{aligned}L^N\big(a(1-\chi)\big)&=|\theta-\xi|^{-2N}\Big(\sum_{k=1}^n(\xi_k-\theta_k)D_{y_k}\Big)^N a\cdot(1-\chi)=\mathcal{O}_N(1)\cdot\frac{(1+|\theta|)^m}{|\theta-\xi|^N}\\&=\mathcal{O}_N(1)\cdot\frac{(1+|\theta|+|\xi|)^{m+n+1-N}}{(1+|\theta|+|\xi|)^{n+1}}=\mathcal{O}_N(1)\cdot\frac{(1+|\xi|)^{m+n+1-N}}{(1+|\theta|)^{n+1}}.\end{aligned}$$

Here we write $\mathcal{O}_N(1)$ for a term uniformly bounded in (ξ,θ), and depending on $N\in\mathbb{N}$ and seminorms of a in $S^m(X\times X\times\mathbb{R}^n)$, we used (4.6) in third equality, and we assume $m+n+1-N<0$. For such $N\in\mathbb{N}$, from (4.5) we get $b_2(x,\xi)=\mathcal{O}_k(1)(1+|\xi|)^{m+n+1-N}$. Similar estimates for derivatives of b_2 yield $b_2\in S^{-\infty}(X\times\mathbb{R}^N)$.

Step 3: method of stationary phase. It remains to study for $|\xi|\geq 1$ and x in a compact set $K\subset X$, the following classical integral (it is only an integral over a compact set):

$$\begin{aligned}b_1(x,\xi)&:=(2\pi)^{-n}\int_X\int_{\mathbb{R}^n}a(x,y,\theta)\chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)e^{i(x-y)\cdot(\theta-\xi)}\,d\theta\,dy\\&=(2\pi)^{-n}\int_{-x+X}\int_{\mathbb{R}^n}a(x,x+s,\xi+\sigma)\chi\Big(\frac{|\sigma|}{|\xi|}\Big)e^{-is\cdot\sigma}\,d\sigma\,ds\\&=\Big(\frac{\lambda}{2\pi}\Big)^n\int_{-x+X}\underbrace{\int_{\mathbb{R}^n}a(x,x+s,\lambda(\omega+\sigma))\chi(|\sigma|)}_{u(s,\sigma):=}e^{-i\lambda s\cdot\sigma}\,d\sigma\,ds,\end{aligned}$$

where in the second line we used the substitutions $y=x+s$ and $\theta=\xi+\sigma$, in the third line we substituted σ by $\lambda\cdot\sigma$, where $\lambda=|\xi|$, and wrote $\xi=\lambda\omega$ where $\omega\in\mathbb{R}^n$ is of unit length. Let us apply Theorem 3.3 to the previous integral. More precisely, use the expansion in (3.8) to write

$$\begin{aligned}b_1(x,\lambda\omega)&=\sum_{|\alpha|\leq N-1}\frac{1}{\alpha!(i\lambda)^{|\alpha|}}\partial_s^\alpha\partial_\sigma^\alpha\Big(a(x,x+s,\lambda(\omega+\sigma))\Big)\Big|_{s=\sigma=0}+S_N(\lambda)\\&=\sum_{|\alpha|\leq N-1}\frac{1}{\alpha!i^{|\alpha|}}\Big(\partial_y^\alpha\partial_\xi^\alpha a(x,y,\xi)\Big)\Big|_{y=x}+S_N(\lambda),\end{aligned}$$

where the remainder term is estimated by Theorem 3.3 as (where the supremum in (s,σ) is over a suitable compact set depending on K and $\Pi_{(x,y)}\operatorname{supp}(a)$):

$$|S_N(\lambda)| \leq C_{N,K}\lambda^{-N} \sum_{|\alpha|+|\beta|\leq 2N+2n+1} \sup_{s,\sigma} \big|\partial_s^\alpha \partial_\sigma^\beta u\big|.$$

It follows that $|S_N(\lambda)| \leq C_N \lambda^{m-N} \leq C_N |\xi|^{m-N}$, for some $C_N > 0$ depending only on a and N (and for $|\xi| \geq 1$). A similar argument applies to provide estimates on x and ξ derivatives of $b_1(x,\xi)$ showing directly that $b_1 \in S^m(X \times \mathbb{R}^n)$, and that $b = b_1 + b_2$ has the right asymptotic expansion (recall that $b_2 \in S^{-\infty}(X \times \mathbb{R}^n)$ by Step 2).

end of Lecture 6, 25.10.2022

Step 4. Finally, for $u \in C_0^\infty(X)$ write $u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{u}(\xi)\, d\xi$ and approximate this integral by finite Riemann sums for $\varepsilon > 0$

$$u_\varepsilon(x) := \Big(\frac{\varepsilon}{2\pi}\Big)^n \sum_{\xi \in (\varepsilon\mathbb{Z})^n, |\xi| \leq \frac{1}{\varepsilon}} e^{ix\cdot\xi}\widehat{u}(\xi),$$

where $u_\varepsilon \to u$ in $C^\infty(X)$ as $\varepsilon \to 0$. Since by Proposition 4.1, $A : C^\infty(X) \to C^\infty(X)$ is continuous, we get using (4.4)

$$Au(x) = \lim_{\varepsilon\to 0} Au_\varepsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} b(x,\xi)\widehat{u}(\xi)\, d\xi,$$

completing the proof. □

By Proposition 4.2, any $A \in \Psi^m(X)$ can be decomposed as $A = A_0 + A_1$, where A_1 is smoothing and A_0 properly supported. Theorem 4.3 then gives a well-defined bijective map

$$\sigma : \Psi^m(X)/\Psi^{-\infty}(X) \to S^m(X\times\mathbb{R}^n)/S^{-\infty}(X\times\mathbb{R}^n), \quad A \mapsto \sigma_{A_0}, \tag{4.7}$$

i.e. the complete symbol map is a well-defined bijection modulo smoothing terms (we emphasise that (4.7) applies to possibly non-properly supported operators). Note that the inverse of this map is the quantisation Op, i.e. $\operatorname{Op}(\sigma_A) = \sigma_{\operatorname{Op}(A)} = A$ modulo smoothing terms.

Let us next discuss adjoint operators. Equip $L^2(X)$ with the standard inner product $(u,v) := \int u(x)\overline{v}(x)\, dx$. Note that $(u,v) = \langle u, \overline{v}\rangle$ if we identify the inner product with the distributional pairing. If $A \in \Psi^m(X)$, its *complex adjoint* $A^* : C_0^\infty(X) \to C^\infty(X)$ is defined by $(Au,v) := (u, A^*v)$, where $u,v \in C_0^\infty(X)$. On the level of Schwartz kernels we have $K_{A^*}(x,y) = \overline{K_A(y,x)}$ and so (similarly to Proposition 2.16, Item 2), we have

$$A^* = \operatorname{Op}(a^*), \qquad a^*(x,y,\xi) = \overline{a(y,x,\xi)}, \tag{4.8}$$

which implies $A^* \in \Psi^m(X)$. Let us record two consequences of Theorem 4.3:

Theorem 4.4. *Let $A \in \Psi^m(X)$. Then:*

1. *The adjoint $A^* \in \Psi^m(X)$ satisfies the asymptotic expansion*

$$\sigma_{A^*}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\sigma_A(x,\xi)}. \tag{4.9}$$

2. *Let $B \in \Psi^{m'}(X)$, with at least one of A, B properly supported. Then the composition $A \circ B \in \Psi^{m+m'}(X)$ is a pseudodifferential operator of order $m + m'$ and*

$$\sigma_{AB}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_A(x,\xi) D_x^\alpha \sigma_B(x,\xi). \tag{4.10}$$

Proof. Item 1. By Proposition 4.2 we may assume A is properly supported. Then Theorem 4.3 implies that $A = \mathrm{Op}(a)$ for some $a(x,\xi) = \sigma_A(x,\xi)$ with $a \in S^m(X \times \mathbb{R}^n)$, so by (4.8) $a^*(x,y,\xi) = a(y,\xi)$ satisfies $A^* = \mathrm{Op}(a^*)$. The expansion (4.9) then follows directly from Theorem 4.3.

Item 2: first proof. Observe firstly that the compositions of a properly supported pseudodifferential and smoothing operators C and D, respectively, CD and DC, are both continuous $\mathcal{E}'(X) \to C^\infty(X)$ by Propositions 2.17 and 4.1, so again by Proposition 2.17 we get $CD, DC \in \Psi^{-\infty}(X)$. By using Proposition 4.2 and observing that the result is unaffected by changes modulo $\Psi^{-\infty}(X)$, this allows us to assume that both A and B are properly supported.

By Theorem 4.3 we have that $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(b)$, where $a = \sigma_A$ and $b = \sigma_B$. Approximate the integral defining Bu by Riemann sums converging in $C^\infty(X)$ as in the proof of Theorem 4.3, Step 4, to obtain

$$ABu = (2\pi)^{-n} A \int_{\mathbb{R}^n} e^{i\bullet\cdot\xi} b(\bullet,\xi)\widehat{u}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \underbrace{e^{-ix\cdot\xi} A\big(e^{i\bullet\cdot\xi} b(\bullet,\xi)\big)}_{c(x,\xi):=} \widehat{u}(\xi)\, d\xi.$$

Following the proof of Theorem 4.3, i.e. studying the oscillatory integral

$$c(x,\xi) = e^{-ix\cdot\xi} A\big(e^{i\bullet\cdot\xi} b(\bullet,\xi)\big)(x) = (2\pi)^{-n} \int_X \int_{\mathbb{R}^n} e^{i(x-y)\cdot(\theta-\xi)} a(x,\theta) b(y,\xi)\, d\theta\, dy \tag{4.11}$$

using the method of stationary phase, we find $c \in S^{m+m'}(X)$ and

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\theta^\alpha D_y^\alpha \big(a(x,\theta) b(y,\xi)\big)\big|_{y=x,\theta=\xi} = \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_A(x,\xi) D_x^\alpha \sigma_B(x,\xi).$$

Step 2: second proof. Alternatively, still assuming both A and B are properly supported, argue as follows. Notice that ${}^t({}^tB) = B$, so by Proposition 2.16, Item 2, and Theorem 4.3, we may write ${}^tB = \mathrm{Op}(\sigma_{{}^tB})$, and so $B = \mathrm{Op}(b)$, where $b(x,y,\xi) = \sigma_{{}^tB}(y,-\xi)$. It follows that for $u \in C_0^\infty(X)$:

$$Bu(x) = (2\pi)^{-n} \int_X e^{ix\cdot\xi} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} \sigma_{{}^tB}(y,-\xi) u(y)\, dy\, d\xi,$$

or equivalently by Fourier inversion $\widehat{Bu}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} \sigma_{{}^tB}(y,-\xi) u(y)\, dy$. Therefore:

$$ABu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_X e^{i(x-y)\cdot\xi} \sigma_A(x,\xi) \sigma_{{}^tB}(y,-\xi) u(y)\, dy\, d\xi,$$

which implies that $AB = \mathrm{Op}(c) \in \Psi^{m+m'}(X)$, where $c(x,y,\xi) = \sigma_A(x,\xi)\sigma_{^tB}(y,-\xi)$. Thus:

$$
\begin{aligned}
\sigma_{AB}(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha\big(\sigma_A(x,\xi)\sigma_{^tB}(y,-\xi)\big)|_{y=x}\\
&\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\big(\sigma_A(x,\xi)D_x^\alpha\sigma_{^tB}(x,-\xi)\big)\\
&\sim \sum_\alpha \frac{1}{\alpha!}\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\partial_\xi^\beta\sigma_A(x,\xi)\cdot\partial_\xi^{\alpha-\beta}D_x^\alpha[\sigma_{^tB}(x,-\xi)]\\
&\sim \sum_\beta \frac{1}{\beta!}\partial_\xi^\beta\sigma_A(x,\xi)\cdot D_x^\beta\sum_{\alpha\ge\beta}\frac{1}{(\alpha-\beta)!}\partial_\xi^{\alpha-\beta}D_x^{\alpha-\beta}[\sigma_{^tB}(x,-\xi)]\\
&\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\sigma_A(x,\xi)D_x^\alpha\sigma_B(x,\xi),
\end{aligned}
$$

where we used Theorem 4.3 in the first and last lines, and that $b(x,y,\xi) = \sigma_{^tB}(y,-\xi)$, completing the proof. □

Remark 4.5. We remark that the composition of two properly supported pseudodifferential operator is again a properly supported pseudodifferential operator, as follows from the proof of Theorem 4.4, Item 2.

4.2. **Changes of variables.** Let us discuss transformations of PDOs under changes of variables. Let $\kappa : X \to \widetilde{X} \subset \mathbb{R}^n$ be a C^∞ diffeomorphism. This defines the pullback κ^* and the pushforward κ_* by

$$\kappa^* : C^\infty(\widetilde{X}) \ni u \mapsto u\circ\kappa \in C^\infty(X), \quad \kappa_* := (\kappa^{-1})^* : C^\infty(X) \to C^\infty(\widetilde{X}).$$

We will also denote by $\kappa'(x) = \frac{\partial\kappa}{\partial x}(x)$ and ${}^t\kappa'(x)$ the differential of κ at x and its transpose, respectively.

Proposition 4.6. *Let $\widetilde{A} = \mathrm{Op}(\widetilde{a}) \in \Psi^m(\widetilde{X})$ for some $\widetilde{a} \in S^m(\widetilde{X}\times\widetilde{X}\times\mathbb{R}^n)$. Then the operator $A := \kappa^*\circ\widetilde{A}\circ\kappa_* : C_0^\infty(X) \to C^\infty(X)$ belongs to $\Psi^m(X)$. Moreover,*

$$\sigma_A(x,\xi) = \sigma_{\widetilde{A}}\big(\kappa(x), \big({}^t\kappa'(x)\big)^{-1}\xi\big) \mod S^{m-1}(X\times\mathbb{R}^n).$$

Proof. Write as an oscillatory integral

$$
\begin{aligned}
Au(x) = \widetilde{A}(u\circ\kappa^{-1})\big(\kappa(x)\big) &= (2\pi)^{-n}\int_{\widetilde{X}}\int_{\mathbb{R}^n} e^{i(\kappa(x)-\widetilde{y})\cdot\widetilde{\xi}}\,\widetilde{a}\big(\kappa(x),\widetilde{y},\widetilde{\xi}\big)u(\kappa^{-1}(\widetilde{y}))\,d\widetilde{\xi}\,d\widetilde{y}\\
&= (2\pi)^{-n}\int_X\int_{\mathbb{R}^n} e^{i\overbrace{(\kappa(x)-\kappa(y))\cdot\widetilde{\xi}}^{\Phi(x,y,\widetilde{\xi}):=}}\underbrace{\widetilde{a}(\kappa(x),\kappa(y),\widetilde{\xi})\Big|\det\frac{\partial\widetilde{y}}{\partial y}\Big|(y)}_{\widetilde{b}(x,y,\widetilde{\xi}):=}\,u(y)\,d\widetilde{\xi}\,dy,
\end{aligned}
$$

where in the last line we changed the variables by $\widetilde{y} = \kappa(y)$ and wrote $\Big|\det\frac{\partial\widetilde{y}}{\partial y}\Big|$ for the Jacobian. Note that $\widetilde{b} \in S^m(X\times X\times\mathbb{R}^n)$ and that $\Phi(x,y,\widetilde{\xi})$ is a phase function, so A is an FIO. By a change of variables in the fibres (usually referred to as the *Kuranishi trick*), we would like to show that in fact $A \in \Psi^m(X)$. By Proposition 2.15, Item 1, we have that the kernel $K_A = (2\pi)^{-n} I(\widetilde{b}, \Phi)$ is smooth away from the diagonal $\Delta(X) \subset X \times X$. Therefore, it suffices to study K_A in a small open neighbourhood Ω of $\Delta(X)$. Set

$$F(x, y) := \int_0^1 \frac{\partial \kappa}{\partial x}\big(tx + (1-t)y\big)\, dt, \qquad (x, y) \in \Omega.$$

It follows that F and its transpose $G := {}^tF$ belong to $C^\infty(\Omega)$, and by observing that $F(x, x) = \frac{\partial \kappa}{\partial x}(x)$ is invertible, by taking Ω small enough we may assume F and G to be invertible. Also, by the chain rule we get $F(x, y)(x - y) = \kappa(x) - \kappa(y)$, and so:

$$\Phi(x, y, \widetilde{\xi}) = \langle F(x, y)(x - y), \widetilde{\xi}\rangle = \langle x - y, G(x, y)\widetilde{\xi}\rangle, \qquad \widetilde{\xi} \in \mathbb{R}^n, \quad (x, y) \in \Omega.$$

After a linear change of variables in the fibres $\mathbb{R}^n$ depending on $(x, y) \in \Omega$, that is, setting $\xi := G(x, y)\widetilde{\xi}$, we may introduce

$$a(x, y, \xi) = \widetilde{a}\big(\kappa(x), \kappa(y), G^{-1}(x, y)\xi\big) \frac{\big|\det \frac{\partial \kappa}{\partial y}(y)\big|}{\big|\det G(x, y)\big|}. \tag{4.12}$$

On Ω we obtain that $K_A(x, y) = (2\pi)^{-n} I\big(a, (x-y)\cdot\xi\big)$, so it remains to determine the symbol class of a. On a compact set $K \subset \Omega$, we have that for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n$:

$$\sup_{(x,y,\xi) \in K \times \mathbb{R}^n} \frac{|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)|}{(1 + |\xi|)^{m - |\gamma|}} \leq C \max_{|\widetilde{\alpha}| + |\widetilde{\beta}| \leq |\alpha| + |\beta|} p_{\kappa(K), (\widetilde{\alpha}, \widetilde{\beta}), \gamma}(\widetilde{a}), \tag{4.13}$$

where we applied the chain rule to (4.12) and used the fact that G is invertible to show the inequality. Here $C = C(\kappa, K, \Omega) > 0$ depends on κ and its derivatives, and we used the notation for a seminorm introduced in (2.5). This shows that $a \in S^m(X \times X \times \mathbb{R}^n)$ and so $A \in \Psi^m(X)$, as required.

Note that by (4.7) we may write $\widetilde{A} = \mathrm{Op}(\sigma_{\widetilde{A}})$ up to smoothing terms, so by Theorem 4.3:

$$\sigma_A(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha \Big(\sigma_{\widetilde{A}}\big(\kappa(x), G(x, y)^{-1}\xi\big) \frac{|\det \frac{\partial \kappa}{\partial y}(y)|}{|\det G(x, y)|}\Big)\Big|_{y = x}, \tag{4.14}$$

and in particular, by identifying the leading order terms we get (4.6). $\square$

Remark 4.7. If we were to consider the analogous classes $\Psi^m_{\rho,\delta}(X)$ obtained by quantising $S^m_{\rho,\delta}(X)$ for arbitrary (ρ, δ), then the previous argument would only work if $\rho + \delta = 1$. The point where the proof breaks for other values of (ρ, δ) is precisely the estimate (4.13) for the newly obtained symbol. To make Theorem 4.3 work we would need $\rho > \delta$.

We now introduce the *principal symbol* of a pseudodifferential operator.

Definition 4.8. *Given $A \in \Psi^m(X)$, the* principal symbol *of A is defined as the image of σ_A in $S^m(X \times \mathbb{R}^n)/S^{m-1}(X \times \mathbb{R}^n)$.*

Let us observe that the principal symbol gives rise to a bijection:

$$\Psi^m(X)/\Psi^{m-1}(X) \to S^m(X \times \mathbb{R}^n)/S^{m-1}(X \times \mathbb{R}^n).$$

Indeed, this is a direct consequence of (4.7).

Finally, we introduce an important sub-class of symbols. Say that a symbol $a \in S^m(X \times \mathbb{R}^n)$ is *classical* and write $a \in S^m_{\mathrm{cl}}(X \times \mathbb{R}^n)$, if there exists $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ near zero, and

$(a_j)_{j=-\infty}^m \in C^\infty\big(X \times (\mathbb{R}^n \setminus 0)\big)$, such that a_j is positively homogeneous of degree j for $|\xi| > 0$ (i.e. $a_j(x, \lambda\xi) = \lambda^j a_j(x, \xi)$ for $\lambda > 0$), such that the following asymptotic expansion holds:

$$a(x,\xi) \sim \sum_{j=0}^{\infty} \big(1 - \chi(\xi)\big) a_{m-j}(x,\xi). \tag{4.15}$$

Recall here that $(1-\chi)a_j \in S^m(X \times \mathbb{R}^n)$ by Example 2.3, Item 2, so the asymptotic sum makes sense by Proposition 2.9. Let $\Psi^m_{\mathrm{cl}}(X) \subset \Psi^m(X)$ denote the class of *classical* pseudodifferential operators, i.e. those $A \in \Psi^m(X)$ such that $\sigma_A \in S^m_{\mathrm{cl}}(X \times \mathbb{R}^n)$. Observe that differential operators are classical pseudodifferential operators.

end of Lecture 7, 1.11.2022

Exercise 4.9. The space $\Psi_{\mathrm{cl}}(X) = \cup_{m \in \mathbb{R}} \Psi_{\mathrm{cl}}(X)$ of classical pseudodifferential operators is closed under composition and invariant under coordinate changes in the sense of Theorem 4.4 and Proposition 4.6, respectively. Moreover, the asymptotic expansion in (4.15) is (essentially) unique, and the principal symbol of $A = \mathrm{Op}(\sigma_A) \in \Psi^m(X)$ for $a := \sigma_A \in S^m_{\mathrm{cl}}(X)$ may be identified with the function $a_m(x,\xi)$ in (4.15).

Proof. Invariances under composition and coordinate change follow directly from respective asymptotic formulas (4.10) and (4.14). Indeed, if the involved complete symbols are classical, then the symbols in the corresponding formulas are also classical.

For the second claim, assume that $\sum_{j=0}^\infty (1-\chi)a_{m-j} \sim \sum_{j=0}^\infty (1-\widetilde{\chi})b_{m-j}$ for some $b_j \in C^\infty\big(X \times (\mathbb{R}^n \setminus 0)\big)$ positively j-homogeneous, and some $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widetilde{\chi} = 1$ near 0. Then we claim that $a_j(x,\xi) \equiv b_j(x,\xi)$. We show this for $j = m$, and the other cases follow by induction. Indeed, by definition of asymptotic summation $c := (1-\chi)a_m - (1-\widetilde{\chi})b_m \in S^{m-1}(X \times \mathbb{R}^n)$, which is positively homogeneous of degree m for $|\xi|$ large enough but $(1+|\xi|)^{-(m-1)}c$ is bounded over compact sets $K \subset X$ uniformly in ξ. This implies that $c \equiv 0$ and proves the claim.

Finally, observe that by definition $a - (1-\chi)a_m \in S^{m-1}_{\mathrm{cl}}(X)$, so the principal symbol map agrees with $(1-\chi)a_m$ which by the uniqueness of (4.15) may be identified with a_m. $\square$

5. Elliptic operators and L^2-continuity

Aim. Using the developed composition calculus, we would like to construct inverses of *elliptic operators*, called *parametrices.* Moreover, we introduce *Sobolev spaces* and study the continuity of pseudodifferential operators.

5.1. Parametrices.

Let $P \in \Psi^m(X)$ with complete symbol $p \in S^m(X \times \mathbb{R}^n)$, i.e. $P = \mathrm{Op}(p)$. Say that P is *elliptic* or *non-characteristic* at $(x_0, \xi_0) \in X \times \big(\mathbb{R}^n \setminus 0\big)$ if there exists a conical neighbourhood V of (x_0,ξ_0) in $X \times \big(\mathbb{R}^n \setminus 0\big)$ and $C > 0$ such that

$$|p(x,\xi)| \geq \frac{1}{C}(1+|\xi|)^m, \quad (x,\xi) \in V, \quad |\xi| \geq C. \tag{5.1}$$

Say that P is *elliptic at* $x_0 \in X$ if P is elliptic at (x_0,ξ) for all $\xi \in \mathbb{R}^n \setminus 0$. Furthermore, we say that P is *elliptic* if it is elliptic at any $x \in X$.

In what follows we denote the symbols by lowercase letters.

Theorem 5.1. *If $P \in \Psi^m(X)$ is elliptic, there exists $Q \in \Psi^{-m}(X)$ properly supported such that*

$$P \circ Q \equiv Q \circ P \equiv \mathrm{Id} \mod \Psi^{-\infty}(X).$$

Moreover, Q is unique modulo $\Psi^{-\infty}(X)$.

Proof. By means of a partition of unity, we may find $q_0 \in C^\infty(X \times \mathbb{R}^n)$ such that for every $K \subset X$ compact, there exists $C_K > 0$ with

$$q_0(x,\xi) = \frac{1}{p(x,\xi)}, \qquad \forall x \in K, |\xi| \geq C_K. \tag{5.2}$$

Indeed, we may exhaust X by compact sets $K_i \nearrow X$ such that $\cup_i K_i^\circ = X$ (where $\bullet^\circ$ denotes the interior of $\bullet$). Let $(\chi_i)_{i=1}^\infty$ be a partition of unity subordinate to this cover. By compactness of K_i and ellipticity of P, for each i there exists $C_i > 0$ such that $p(x,\xi)$ satisfies (5.1) for $|\xi| \geq C_i$ and $x \in K_i$. Let $\psi_i \in C_0^\infty(X)$ such that $\psi_i(\xi) = 1$ for $|\xi| \leq C_i$. Then it is straightforward to see that

$$q_0(x,\xi) := \sum_i \chi_i(x) p^{-1}(x,\xi)\big(1 - \psi_i(\xi)\big)$$

satisfies the required conditions. Moreover, we have:

Lemma 5.2. *It holds that $q_0(x,\xi) \in S^{-m}(X \times \mathbb{R}^n)$.*

Proof. Let $K \subset X$ be compact. By (5.2), there exists $D > 0$ such that for $x \in K$ with $|\xi| \geq D$, we have $|q_0(x,\xi)| \leq \frac{1}{D}(1+|\xi|)^{-m}$. This shows the initial bound in the symbol estimate (2.2). To show the other ones by induction we may assume for $x \in K$ and $|\alpha| + |\beta| < N$ that

$$|\partial_x^\alpha \partial_\xi^\beta q_0(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{-m-|\beta|}. \tag{5.3}$$

Now assume $|\alpha| + |\beta| = N$. Differentiating $q_0(x,\xi)p(x,\xi) = 1$ for $x \in K$ and $|\xi| \geq D$, we get

$$|p\partial_x^\alpha \partial_\xi^\beta q_0| \leq \sum_{\substack{\alpha' \leq \alpha,\, \beta' \leq \beta \\ (\alpha',\beta') \neq (\alpha,\beta)}} \big|C(\alpha',\beta',\alpha,\beta)\big|\big|\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} p\big|\big|\partial_x^{\alpha'}\partial_\xi^{\beta'} q_0\big| \leq C(1+|\xi|)^{-|\beta|},$$

for some $C > 0$, where $C(\alpha',\beta',\alpha,\beta)$ is a suitable multinomial coefficient, and we used the symbol estimates for p and (5.3). Combining with the ellipticity assumption on p, this proves the required bound on $\partial_x^\alpha \partial_\xi^\beta q_0$ and completes the proof. □

By Lemma 5.2, $\mathrm{Op}(q_0) \in \Psi^m(X)$, so we may take $Q_0 \in \Psi^{-m}(X)$ properly supported with $Q_0 - \mathrm{Op}(q_0) \in \Psi^{-\infty}(X)$. By Theorem 4.4 we have $Q_0 \circ P - \mathrm{Id} \in \Psi^{-1}(X)$. Similarly, there exists (properly supported) $Q_1 \in \Psi^{-m-1}(X)$ such that

$$(Q_0 + Q_1) \circ P - \mathrm{Id} \in \Psi^{-2}(X).$$

Indeed, by Theorem 4.4 we see it suffices to take $q_1 = -q_0(\sigma_{Q_0 P} - 1) \in S^{-m-1}(X \times \mathbb{R}^n)$ and so $\mathrm{Op}(q_1) \in \Psi^{-m-1}(X)$. Iterating, we obtain Q_k with $Q_k - \mathrm{Op}(q_k) \in \Psi^{-\infty}(X)$ for all $k \in \mathbb{N}_0$, such that

$$(Q_0 + Q_1 + \ldots + Q_k) \circ P - \mathrm{Id} \in \Psi^{-k-1}(X).$$

Choosing Q_L properly supported to be equal to $\mathrm{Op}\big(\sum_{j\geq 0} q_j\big)$ up to smoothing gives $Q_L \circ P - \mathrm{Id} \in \Psi^{-\infty}(X)$. Arguing similarly, there is a properly supported Q_R with $P \circ Q_R - \mathrm{Id} \in \Psi^{-\infty}(X)$. Applying Q_R to the right of $Q_L P \equiv \mathrm{Id} \mod \Psi^{-\infty}(X)$ gives $Q := Q_L \equiv Q_R \mod \Psi^{-\infty}(X)$. Uniqueness of Q similarly follows, which completes the proof. □

The Q constructed in Theorem 5.1 is called the (elliptic) *parametrix* of P.

Corollary 5.3. *Assume $P \in \Psi^m(X)$ is elliptic and properly supported. Then:*

1. *$P : \mathcal{D}'(X)/C^\infty(X) \to \mathcal{D}'(X)/C^\infty(X)$ is a bijection;*

2. *For $u \in \mathcal{D}'(X)$, we have* $\operatorname{sing\,supp}(Pu) = \operatorname{sing\,supp}(u)$. *In particular, we obtain the* elliptic regularity *result: if $u \in \mathcal{D}'(X)$ is such that $Pu \in C^\infty(X)$, then $u \in C^\infty(X)$.*

Proof. Note that P is well-defined on $\mathcal{D}'(X)$ by Proposition 4.1. For Item 1, simply observe that the inverse of P is given by the parametrix Q from Theorem 5.1.

For Item 2, $\operatorname{sing\,supp}(Pu) \subset \operatorname{sing\,supp}(u)$ by pseudolocality, see Proposition 2.16, Item 5. For the other inclusion, applying Theorem 5.1 gives $Q \in \Psi^{-m}(X)$ such that $QPu - u \in C^\infty(X)$, which implies the claim again by pseudolocality. □

Remark 5.4. Let us make a couple of remarks at this point:

1. More generally, a PDO P is *hypoellipic* if $Pu \in C^\infty(X)$ implies $u \in C^\infty(X)$. It is possible to show, for example, that the *heat operator* $\partial_t - \Delta$ is hypoelliptic (but clearly *not* elliptic).
2. Later we will be able to show a more precise regularity result using *wavefront sets.*

5.2. **L^2-continuity.** We next focus on mapping properties of operators in $\Psi^0(X)$ between L^2 spaces at first, and then generalise to mappings on Sobolev spaces.

Theorem 5.5. *Let $A \in \Psi^0(\mathbb{R}^n)$ with $K_A \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$. Define*

$$M := \limsup_{|\xi|\to\infty} \sup_{x\in\mathbb{R}^n} |\sigma_A(x,\xi)|.$$

Then $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous and for every $\varepsilon > 0$, we can find a decomposition $A = A_\varepsilon + K_\varepsilon$, where $\|A_\varepsilon\|_{L^2\to L^2} \leq M + \varepsilon$ and $K_\varepsilon \in \Psi^{-\infty}(\mathbb{R}^n)$. Also, we have $\operatorname{supp} K_{A_\varepsilon} \subset \operatorname{supp} K_A + \{0\} \times \mathbb{B}(0,\varepsilon)$.

Proof. Step 1: well-definedness and the idea. Since $\operatorname{supp} K_A$ is compact, A is properly supported and we may write $A = \operatorname{Op}(\sigma_A)$; in fact, $\sigma_A(x,\xi)$ vanishes for x outside a compact set. More precisely, if $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi(x)K_A(x,y) \equiv K_A(x,y)$, then $\operatorname{Op}\big((1-\chi)\sigma_A\big) = 0$ which implies $(1-\chi)\sigma_A = 0$. Therefore M is well-defined. For the L^2-boundedness, use the following idea: show that $A^*A \leq (M+\varepsilon)^2 \operatorname{Id}$ in the sense of self-adjoint operators by constructing a square root of $(M+\varepsilon)^2 \operatorname{Id} - A^*A$. Note that A^*A also has a compactly supported Schwartz kernel (which follows from the formula 4.11 for the symbol of the composition).

Step 2: the 'square root'. The following lemma constructs a square root of $(M+\varepsilon)^2 \operatorname{Id} - A^*A$ up to smoothing.

Lemma 5.6. *For all $\varepsilon > 0$, there exist $B \in \Psi^0(\mathbb{R}^n)$ and $R \in \Psi^{-\infty}(\mathbb{R}^n)$ with $K_{(M+\varepsilon)\operatorname{Id} - B} \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$ and such that*

$$(M+\varepsilon)^2 \operatorname{Id} = A^*A + B^*B + R.$$

Proof. Consider the self adjoint operator $C := (M+\varepsilon)^2 \operatorname{Id} - A^*A$. We claim that we can find $b_0 \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, with $b_0 = M + \varepsilon$ for x outside of a compact set, such that

$$\sigma_C - |b_0|^2 \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, since σ_A is compactly supported in the x variable, there is an $L > 0$ such that $M + \frac{\varepsilon}{2} > |\sigma_A(x,\xi)|$ for all $x \in \mathbb{R}^n$ and $|\xi| > L$. Taking $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 0$ for $|\xi| \geq 2L$ and $\chi(\xi) = 1$ for $|\xi| < L$, we may set

$$b_0(x,\xi) := (1-\chi(\xi))\sqrt{(M+\varepsilon)^2 - |\sigma_A(x,\xi)|^2} + \chi(\xi)(M+\varepsilon). \tag{5.4}$$

One checks that $b_0 \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ similarly to the proof of Lemma 5.2. Moreover, by Theorem 4.4 we have $\sigma_{A^*A} - |\sigma_A|^2 \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$, so

$$\sigma_C - |b_0|^2 \equiv (M+\varepsilon)^2 - |\sigma_A|^2 - |b_0|^2 \equiv 0 \mod S^{-1}(\mathbb{R}^n \times \mathbb{R}^n),$$

where in the last equivalence we used (5.4), proving the claim.

Write $B_0 := \mathrm{Op}(b_0) = (M+\varepsilon)\,\mathrm{Id} + D$, where $\operatorname{supp} K_D$ is compact and $D \in \Psi^0(\mathbb{R}^n)$. Therefore:

$$\Psi^{-1}(\mathbb{R}^n) \ni C_0 := C - B_0^* B_0 = -A^*A - (M+\varepsilon)(D^* + D) - D^*D,$$

where $\operatorname{supp} K_{C_0}$ is compact and so σ_{C_0} is compactly supported in x. Next, we seek for the next approximation $B_0 + B_1$, for some $B_1 \in \Psi^{-1}(\mathbb{R}^n)$ such that:

$$\Psi^{-2}(\mathbb{R}^n) \ni C_1 := C - (B_0^* + B_1^*)(B_0 + B_1) = (C - B_0^* B_0) - (B_0^* B_1 + B_1^* B_0) - B_1^* B_1. \quad (5.5)$$

To reduce the order of the operator on the right hand side, define the symbol b_1 by $b_1 := \frac{1}{2} b_0^{-1} \sigma_{C - B_0^* B_0}$. Note here that b_0 is bounded from below by a positive constant by its definition (5.4) and so by Lemma 5.2, $b_0^{-1} \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, implying $b_1 \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, taking B_1 properly supported such that $B_1 - \mathrm{Op}(b_1)$ is smoothing shows (5.5).

Iterate this procedure and obtain $B_k \in \Psi^{-k}(\mathbb{R}^n)$ with symbols $b_k \in S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$C - (B_0 + \ldots + B_k)^*(B_0 + \ldots + B_k) \in \Psi^{-k-1}(\mathbb{R}^n).$$

The operator B is then defined to be $\mathrm{Op}\big(\sum_{k=0}^\infty b_k\big)$, proving the lemma. $\square$

Step 3: the L^2-bound. For $u \in C_0^\infty(\mathbb{R}^n)$ and the operator $B \in \Psi^0(\mathbb{R}^n)$ constructed in Lemma 5.6, we have: $(B^*Bu, u) = \|Bu\|_{L^2}^2 \geq 0$. Thus

$$0 \leq \|Bu\|_{L^2}^2 = (B^*Bu, u) = (M+\varepsilon)^2 \|u\|_{L^2}^2 - \|Au\|_{L^2}^2 - (Ru, u). \quad (5.6)$$

Note by Lemma 5.6 that $R \in \Psi^{-\infty}(\mathbb{R}^n)$ and the Schwartz kernel of R is compactly supported, so $R : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded, which combined with (5.6) shows the continuity property.

Step 4: the sharp estimate. For the second part, let $\psi \in C_0^\infty\big(\mathbb{B}(0, \frac{1}{2})\big)$ with $\int \psi(x)\,dx = 1$ and $\psi \geq 0$. Then also $0 \leq |\widehat{\psi}| \leq 1$ and $\widehat{\psi}(0) = 1$. Put $\chi = \psi * \check{\psi} \in C_0^\infty\big(\mathbb{B}(0,1)\big)$, where $\check{\psi}(x) := \psi(-x)$. Then $\widehat{\chi} = |\widehat{\psi}|^2$ by Proposition 1.19, Item 3, so also $0 \leq \widehat{\chi} \leq 1$ and $\widehat{\chi}(0) = 1$. Put $\chi_\varepsilon(x) := \varepsilon^{-n}\chi\big(\frac{x}{\varepsilon}\big)$ and define $P_\varepsilon u := u - \chi_\varepsilon * u$.

Now by Proposition 1.19, Items 3 and 7, we get

$$\|P_\varepsilon u\|_{L^2}^2 = (2\pi)^{-n} \|\widehat{u}(1 - \widehat{\psi})\|_{L^2}^2 \leq (2\pi)^{-n} \|\widehat{u}\|_{L^2}^2 = \|u\|_{L^2}^2.$$

Thus applying the inequality (5.6) to $P_\varepsilon u$ we conclude that

$$\|AP_\varepsilon u\|_{L^2}^2 \leq (M+\varepsilon)^2 \|u\|_{L^2}^2 + \|RP_\varepsilon u\|_{L^2} \|u\|_{L^2}.$$

Observe that the Schwartz kernel K_ε of RP_ε is equal to:

$$K_\varepsilon(x,y) = K_R(x,y) - \int_{\mathbb{R}^n} K_R(x,z)\chi_\varepsilon(z-y)\,dz,$$

which converges uniformly to zero in $C^\infty(\mathbb{R}^n)$ as $\varepsilon \to 0$. It follows that for any $\varepsilon_0 > 0$, for $\varepsilon > 0$ small enough we have $\|AP_\varepsilon u\|_{L^2} \leq (M + \varepsilon_0)\|u\|_{L^2}^2$. Finally, define $A_\varepsilon := AP_{\varepsilon'}$ and $K_\varepsilon(\bullet) := A(\chi_{\varepsilon'} * \bullet) \in \Psi^{-\infty}(\mathbb{R}^n)$, for some ε' small enough depending on ε. This completes the proof. $\square$

---

end of Lecture 8, 8.11.2022

5.3. **Sobolev spaces and PDOs.** Let us now extend the L^2-continuity of PDOs to properties about mappings between *Sobolev spaces*.

For each $s \in \mathbb{R}$, we define the *Sobolev space* $H^s(\mathbb{R}^n)$ as

$$H^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \in L^2(\mathbb{R}^n)\},$$

equipped with the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} := (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi,$$

with the associated inner product given explicitly by

$$(u,v)_{H^s(\mathbb{R}^n)} := (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi.$$

Note that for $s = 0$ we have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ (using Proposition 1.19, Item 7). Let us collect some basic properties of Sobolev spaces in one proposition:

Proposition 5.7 (Properties of Sobolev spaces)**.**

1. *$H^s(\mathbb{R}^n)$ is a Hilbert space with the inner product given by (5.3) and $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ is dense;*
2. *$H^s(\mathbb{R}^n) \subset H^t(\mathbb{R}^n)$ for $s > t$. Moreover, this inclusion is compact;*
3. *There exists a non-degenerate continuous pairing, continuously extending the one on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$*
$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \to \mathbb{C}, \quad (u,v) := \int_{\mathbb{R}^n} u(x)v(x)\,dx. \tag{5.7}$$
This gives a canonical isometric isomorphism $H^{-s}(\mathbb{R}^n) = \big(H^s(\mathbb{R}^n)\big)'$, the dual space of $H^s(\mathbb{R}^n)$;
4. *For $m \in \mathbb{N}$, there is an alternative definition of Sobolev spaces:*
$$H^m(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) \mid D^\alpha u \in L^2(\mathbb{R}^n), \ \forall |\alpha| \le m\}. \tag{5.8}$$
The norm $\|u\|_{H^m}$ is equivalent to the norm $\sum_{|\alpha|\le m} \|D^\alpha u\|_{L^2(\mathbb{R}^n)}$;
5. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}$. Then the multiplication map $M_\varphi : \mathcal{S}(\mathbb{R}^n) \ni u \mapsto \varphi u \in \mathcal{S}(\mathbb{R}^n)$, extends to a continuous map $M_\varphi : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$. Therefore a differential operator P of order m with constant coefficients or coefficients in $\mathcal{S}(\mathbb{R}^n)$ is continuous $P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$;*
6. *For $s, \ell \in \mathbb{R}$, the operator $(1-\Delta)^{\frac{\ell}{2}} := \mathrm{Op}\big((1+|\xi|^2)^{\frac{\ell}{2}}\big) : H^s(\mathbb{R}^n) \to H^{s-\ell}(\mathbb{R}^n)$ is an isometric isomorphism;*
7. *Sobolev embedding theorem holds: for any $k \in \mathbb{N}_0$, if $s > \frac{n}{2} + k$, then $H^s(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$ continuously.*

Proof. The proofs mostly follow [3, Chapter 9.3].

Item 1. Note that $(\bullet,\bullet)_{H^s}$ is well-defined by Cauchy-Schwarz, and that $\|u\|_{H^s} = 0$ implies that $\widehat{u} = 0$ as a distribution in $\mathcal{S}'(\mathbb{R}^n)$, which by Proposition 1.19, Item 6, implies $u = 0$. Thus $\|\bullet\|_{H^s}$ is positive-definite. To show it is complete, assume $(u_i)_{i=1}^\infty \subset H^s(\mathbb{R}^n)$ is Cauchy. Then by definition $\big((1+|\xi|^2)^{\frac{s}{2}}\widehat{u}_i\big)_{i=1}^\infty$ is Cauchy in $L^2(\mathbb{R}^n)$, and by completeness of $L^2(\mathbb{R}^n)$ we get $(1+|\xi|^2)^{\frac{s}{2}}\widehat{u}_i \to v$ in $L^2(\mathbb{R}^n)$. Writing $v = (1+|\xi|^2)^{\frac{s}{2}}\widehat{u}$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$ gives $u_i \to u$ in $H^s(\mathbb{R}^n)$ and proves completeness of $H^s(\mathbb{R}^n)$.

For the density part, let $u \in H^s(\mathbb{R}^n)$. By definition $(1+|\xi|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^n)$ so by Exercise 1.20 there exist $(\phi_j)_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ such that $\phi_j \to (1+|\xi|^2)^{\frac{s}{2}}\widehat{u}$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$. Writing $\phi_j = (1+|\xi|^2)^{\frac{s}{2}}\widehat{\psi}_j$ for some $\psi_j \in \mathcal{S}(\mathbb{R}^n)$ (possible by Proposition 1.19, Item 6), this immediately implies $\psi_j \to u$ in $H^s(\mathbb{R}^n)$.

Item 2. If $u \in \mathcal{S}(\mathbb{R}^n)$, it is straightforward to see that $\|u\|_{H^t} \leq \|u\|_{H^s}$. The result then follows by density of $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ from the preceding item. The compactness claim is known as the Rellich-Kondrachov theorem.

Item 3. Given $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, the pairing (5.7) is well-defined and satisfies

$$|(\phi,\psi)| = (2\pi)^{-n}\Big|\int_{\mathbb{R}^n}(1+|\xi|^2)^{\frac{s}{2}}\widehat{\phi}(\xi)\cdot(1+|\xi|^2)^{-\frac{s}{2}}\widehat{\psi}(-\xi)\,d\xi\Big| \leq \|\phi\|_{H^s}\|\psi\|_{H^{-s}}, \tag{5.9}$$

where we used Proposition 1.19, Item 7, and Cauchy-Schwarz inequality. This implies that the pairing extends continuously to $H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n)$. To show the isometry claim, let $u \in H^{-s}(\mathbb{R}^n)$. By (5.9), the map $H^s(\mathbb{R}^n) \ni v \mapsto (v,u) \in \mathbb{C}$ is a continuous linear functional with norm bounded by $\|u\|_{H^{-s}}$. In fact, taking $v_0 := \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s}\widehat{u}(-\xi)\big) \in H^s(\mathbb{R}^n)$, by definition we get $(u, v_0) = \|u\|_{H^{-s}}^2$, which shows that $H^{-s}(\mathbb{R}^n) \ni u \mapsto (\bullet, u) \in \big(H^s(\mathbb{R}^n)\big)'$ is an isometry onto its range.

To show surjectivity, let $u' \in \big(H^s(\mathbb{R}^n)\big)'$. By the Riesz representation theorem, there exists $w \in H^s(\mathbb{R}^n)$ such that

$$u'(v) = (v,w)_{H^s} = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s\widehat{v}(\xi)\overline{\widehat{w}(\xi)}\,d\xi.$$

Setting $u := \mathcal{F}^{-1}\big((1+|\xi|^2)^s\overline{\widehat{w}(-\xi)}\big)$, it follows that $u \in H^{-s}(\mathbb{R}^n)$ and so similarly to the first equality of (5.9) we get $u'(v) = (v,u)$, showing surjectivity and completing the proof.

Item 4. Let us first observe that for a vector $\xi \in \mathbb{R}^n$ and a multi-index α we have $|\xi^\alpha|^2 \leq |\xi|^{2|\alpha|}$, and that also any k

$$|\xi|^{2k} = \Big(\sum_{i=1}^n \xi_i^2\Big)^k = \sum_{|\beta|=k}\frac{k!}{\beta!}|\xi^\beta|^2.$$

It follows that there exists $C > 1$ depending on m such that

$$\frac{1}{C}\sum_{|\alpha|\leq m}|\xi^\alpha|^2 \leq (1+|\xi|^2)^m \leq C\sum_{|\alpha|\leq m}|\xi^\alpha|^2. \tag{5.10}$$

If $u \in \mathcal{S}(\mathbb{R}^n)$, then multiplying (5.10) by $|\widehat{u}|^2$ and integrating yields

$$\frac{1}{C}\sum_{|\alpha|\leq m}\|D^\alpha u\|_{L^2}^2 \leq \|u\|_{H^m}^2 \leq C\sum_{|\alpha|\leq m}\|D^\alpha u\|_{L^2}^2. \tag{5.11}$$

To show (5.11) for $u \in H^m(\mathbb{R}^n)$, by density of $\mathcal{S}(\mathbb{R}^n)$ it suffices to show that if $\mathcal{S}(\mathbb{R}^n) \ni v_i \to v$ in $H^m(\mathbb{R}^n)$ as $i \to \infty$, then $D^\alpha v_i \to D^\alpha v$ in $L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$. However by Item 2 we have $v \in L^2(\mathbb{R}^n)$ and by (5.11) that $(D^\alpha v_i)_{i=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, so $D^\alpha v_i \to v_\alpha$ for some $v_\alpha \in L^2(\mathbb{R}^n)$. By uniqueness of limits, we get $D^\alpha v = v_\alpha$, proving the claim. The same argument shows that the space on the right hand side of (5.9) is equal to $H^m(\mathbb{R}^n)$, and therefore (5.11) gives the equivalence of norms.

Item 6. Note that $(1-\Delta)^{\frac{\ell}{2}} \in \Psi^\ell(\mathbb{R}^n)$ and also by definition

$$(1-\Delta)^{\frac{\ell}{2}} u = \mathcal{F}^{-1}\big((1+|\xi|^2)^{\frac{\ell}{2}}\widehat{u}\big), \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{5.12}$$

This precisely means that $(1-\Delta)^{\frac{\ell}{2}} : H^s(\mathbb{R}^n) \to H^{s-\ell}(\mathbb{R}^n)$ is an isometric isomorphism onto range (here we use the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ established by Item 1 above). The surjectivity is obtained by noting that the inverse map is given by $(1-\Delta)^{-\frac{\ell}{2}}$.

Item 7. Firstly, consider the case $k=0$ and note that $\xi \mapsto (1+|\xi|^2)^{-s} \in L^1(\mathbb{R}^n)$ for $s > \frac{n}{2}$. If $u \in H^s(\mathbb{R}^n)$, then by Cauchy-Schwarz

$$\|\widehat{u}\|_{L^1(\mathbb{R}^n)} \leq \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} \, d\xi\Big)^{\frac{1}{2}} = C(s)\|u\|_{H^s},$$

for some $C(s) > 0$. By the inverse Fourier transform (see Proposition 1.19, Items 1 and 6), we have $u \in C^0(\mathbb{R}^n)$ and $|u(x)| \leq C\|u\|_{H^s}$ for some $C > 0$ and any $x \in \mathbb{R}^n$, proving the claim.

For an arbitrary $k > 0$, apply the preceding paragraph to $D^\alpha u$ for $|\alpha| \leq k$ and note that $D^\alpha u \in H^{s-k}(\mathbb{R}^n)$ for such range of α. This completes the proof. □

We next introduce a few versions of Sobolev spaces for an open set $X \subset \mathbb{R}^n$. The space of distributions that *locally* belong to a Sobolev space is given by

$$H^s_{\mathrm{loc}}(X) = \{u \in \mathcal{D}'(X) \mid \varphi u \in H^s(\mathbb{R}^n), \quad \forall \varphi \in C_0^\infty(X)\}.$$

The space $H^s_{\mathrm{loc}}(X)$ is equipped with the topology of a locally convex space induced by $\mathcal{P} = \{p_\varphi \mid \varphi \in C_0^\infty(X)\}$, where $p_\varphi(u) := \|\varphi u\|_{H^s}$. Then:

Exercise 5.8. *It suffices to take a countable, exhausting family of seminorms, i.e.* $(\varphi_j)_{j=1}^\infty \subset C_0^\infty(X)$ *such that* $\varphi_{j+1} = 1$ *on* $\mathrm{supp}(\varphi_j)$ *and* $\cup_j\{\varphi_j = 1\} = X$. *Then* $H^s_{\mathrm{loc}}(X)$ *becomes a Fréchet space.*

Next, let $K \subset X$ be compact. Define

$$H^s(K) := \{u \in H^s(\mathbb{R}^n) \mid \mathrm{supp}(u) \subset K\} \subset H^s(\mathbb{R}^n).$$

The space $H^s(K)$ inherits the subspace topology, making it into a Hilbert space. Finally, the Sobolev space of distributions with compact support is introduced by

$$H^s_{\mathrm{comp}}(X) = \bigcup_{K \subset X} H^s(K),$$

where the union is over all $K \subset X$ compact. Topologise $H^s_{\mathrm{comp}}(X)$ with the locally convex inductive limit topology, analogous to the topology on test functions $C_0^\infty(X)$ in §1.2.

Exercise 5.9. *The pairing* $H^s_{\mathrm{loc}}(X) \times H^{-s}_{\mathrm{comp}}(X)$ *is non-degenerate, similarly to Proposition 5.7, Item 3.*

We are ready to present the main theorem of this subsection:

Theorem 5.10. *Let* $A \in \Psi^m(X)$ *be properly supported and let* $s \in \mathbb{R}$. *Then* A *is continuous as a map:*

$$A : H^s_{\mathrm{loc}}(X) \to H^{s-m}_{\mathrm{loc}}(X), \quad A : H^s_{\mathrm{comp}}(X) \to H^{s-m}_{\mathrm{comp}}(X).$$

If A *is elliptic, then for all* $u \in \mathcal{D}'(X)$, *we have the* qualitative *elliptic regularity*

$$u \in H^s_{\mathrm{loc}}(X) \iff Au \in H^{s-m}_{\mathrm{loc}}(X).$$

Proof. The last claim follows from the mapping properties and the existence of an elliptic parametrix, Theorem 5.1. Let us give only the argument for the mapping property on the $H^s_{\mathrm{loc}}(X)$ space – the other case follows similarly.

Step 1: reduction to $X = \mathbb{R}^n$. Let $\varphi \in C_0^\infty(X)$. Then it suffices to show $\varphi A : H^s_{\mathrm{loc}}(X) \to H^{s-m}_{\mathrm{loc}}(\mathbb{R}^n)$ is continuous. There exists $\psi \in C_0^\infty(X)$ equal to one on large compact subset of X (more precisely, equal to one on $\Pi_x\big(\operatorname{supp}(\varphi K_A)\big)$, where we use the notation from Section 4), such that $\varphi A = \varphi A \psi$. It suffices to prove $\widetilde{A} = \varphi A \psi : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is continuous. Here $\widetilde{A} \in \Psi^m(\mathbb{R}^n)$ and $\operatorname{supp} K_{\widetilde{A}}$ is compact.

Step 2: isomorphism to L^2. We claim that for every $\ell \in \mathbb{R}$, there is an elliptic, properly supported $\Lambda_\ell \in \Psi^\ell(\mathbb{R}^n)$, such that for all $t \in \mathbb{R}$ we have isomorphisms

$$\Lambda_\ell : H^t(\mathbb{R}^n) \to H^{t-\ell}(\mathbb{R}^n).$$

To prove the claim, let us modify the operator $(1-\Delta)^{\frac{\ell}{2}} \in \Psi^\ell(\mathbb{R}^n)$ introduced in Proposition 5.7, Item 6, into a properly supported $\Lambda_\ell \in \Psi^\ell(\mathbb{R}^n)$ that satisfies the required properties.

First observe that we may write $(1-\Delta)^{\frac{\ell}{2}} u = \mathcal{F}^{-1}(\widehat{v_\ell}\widehat{u})$ for any $u \in C_0^\infty(\mathbb{R}^n)$, where

$$v_\ell(x) = \mathcal{F}^{-1}\big((1+|\xi|^2)^{\frac{\ell}{2}}\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{\ell}{2}}\, d\xi \quad \in \mathcal{S}'(\mathbb{R}^n),$$

understood as an oscillatory integral. By Proposition 2.15, Item 1, we have that v_ℓ is smooth outside $x = 0$. In fact, we claim that for $x \neq 0$, v_ℓ is of Schwartz class (in the sense that $(1-\chi)v_\ell \in \mathcal{S}(\mathbb{R}^n)$ for $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ close to zero) since for any $k \in \mathbb{N}$:

$$v_\ell(x) = (2\pi)^{-n} \frac{1}{|x|^{2k}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(-\Delta_\xi)^k (1+|\xi|^2)^{\frac{\ell}{2}}\, d\xi,$$

and similarly for any of the x derivatives of v_ℓ. Take $\chi_\ell \in C_0^\infty(\mathbb{R}^n)$ with $\chi_\ell = 1$ near zero and define $w_\ell := \chi_\ell v_\ell$. Then, as $(1-\chi_\ell)v_\ell \in \mathcal{S}(\mathbb{R}^n)$

$$R_\ell := \widehat{w}_\ell(\xi) - (1+|\xi|^2)^{\frac{\ell}{2}} = \mathcal{F}\big((\chi_\ell - 1)v_\ell\big) \quad \in \mathcal{S}(\mathbb{R}^n). \tag{5.13}$$

Define $\Lambda_\ell u := \mathcal{F}^{-1}\big(\widehat{w}_\ell \widehat{u}\big)$. By (5.13) we have that $\Lambda_\ell \in \Psi^\ell(\mathbb{R}^n)$ and that $\Lambda_\ell : H^t(\mathbb{R}^n) \to H^{t-\ell}(\mathbb{R}^n)$ is continuous. Moreover, Λ_ℓ is properly supported as its Schwartz kernel is $w_\ell(x-y)$ and w_ℓ is compactly supported. Finally, observe that by taking χ_ℓ equal to 1 on a sufficiently large set, we have $\|\mathrm{Op}(R_\ell)\|_{H^t \to H^{t-\ell}} \leq \frac{1}{2}$ for any $t \in \mathbb{R}$. Since $(1-\Delta)^{\frac{\ell}{2}}$ is an isometric isomorphism, by standard functional analysis this shows that Λ_ℓ is an isomorphism, proving the claim.

Step 3: conclusion. By Step 2, note that $H^s(\mathbb{R}^n) = \Lambda_{-s} L^2(\mathbb{R}^n)$. So it suffices to prove that $B := \Lambda_{s-m} \widetilde{A} \Lambda_{-s} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous. But this follows from Theorem 5.5, as $B \in \Psi^0(\mathbb{R}^n)$ and $\operatorname{supp} K_B$ compact. □

---

end of Lecture 9, 15.11.2022

Remark 5.11. If $A \in \Psi^m(X)$ is not assumed properly supported, then it can be shown that $A : H^s_{\mathrm{comp}}(X) \to H^{s-m}_{\mathrm{loc}}(X)$ is continuous. This should be compared to the mapping properties of PDOs in Proposition 2.16, Item 1, i.e. we need compact support of functions in the domain of a general PDO and we only get local control in the range.

Corollary 5.12 (Quantitative elliptic estimates)**.** *Let A be an elliptic differential operator of order m with smooth coefficients. Then for any $K \subset X$ compact, $s \in \mathbb{R}$, and $N \in \mathbb{Z}$, there exists $C = C_{K,s,N} > 0$ such that*

$$\|u\|_{H^{s+m}} \leq C(\|Au\|_{H^s} + \|u\|_{H^{-N}}), \quad \forall u \in H^{s+m}(K).$$

Proof. Let $B \in \Psi^{-m}(X)$ be the parametrix of A. Then $u = BAu + Ru$, where $R \in \Psi^{-\infty}(X)$ is smoothing. The estimate follows by applying Theorem 5.10. □

6. The wavefront set of a distribution

Aim. Using the Fourier transform, refine the notion of singular support by introducing the *wavefront set*, the set of directions where the Fourier transform is *not* rapidly decaying.

Let us use the notation $T^*X = X \times \mathbb{R}^n$ for the cotangent space of some open $X \subset \mathbb{R}^n$ and $0 := X \times \{0\}$ for its zero section. All PDOs in this section are assumed to be properly supported, unless otherwise stated. Recall that if $u \in \mathcal{D}'(X)$, then $x_0 \notin \operatorname{sing\,supp}(u)$ if and only if there is a $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$, such that $\varphi u \in C^\infty(X)$.

Definition 6.1. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \in T^*X \setminus 0$. We say that u is C^∞* microlocally near *(x_0, ξ_0) if there exists $A \in \Psi^m(X)$ elliptic at (x_0, ξ_0), such that $Au \in C^\infty(X)$. Then*

$$\mathrm{WF}(u) := \{(x, \xi) \in T^*X \setminus 0 \mid u \text{ is not } C^\infty \text{ near } (x, \xi)\}.$$

Remark 6.2. We make some initial observations:

1. We may always assume $m = 0$ in Definition 6.1, by composing on the left with $B \in \Psi^{-m}(X)$ elliptic at (x_0, ξ_0);
2. The set of points in $T^*X \setminus 0$ where u is C^∞ is an open conic set, so $\mathrm{WF}(u)$ is a *closed conic set* in $T^*X \setminus 0$;
3. We may introduce *symbols in a cone*. Let $V \subset T^*X \setminus 0$ be any open conic subset. Then $a \in S^m(V)$ if $a \in C^\infty(V)$, and for all $\alpha, \beta \in \mathbb{N}_0^n$ and closed conic sets $W \Subset V$ (in the sense that $W \cap \{|\xi| = 1\}$ is compactly contained in $V \cap \{|\xi| = 1\}$), there exists $C = C(\alpha, \beta, W) > 0$ such that $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C(1 + |\xi|)^{m - |\beta|}$ for $(x, \xi) \in W$. The space $S^{-\infty}(V)$ is defined as $\cap_{m \in \mathbb{R}} S^m(V)$. The general results for symbols from Section 2 remain valid for $S^m(V)$ (products, asymptotic summation).

Definition 6.3. *Given $A \in \Psi^m(X)$, we define its* wavefront set $\mathrm{WF}(A)$ *as the smallest closed cone $\Gamma \subset T^*X \setminus 0$ such that $\sigma_A|_{\Gamma^c} \in S^{-\infty}(\Gamma^c)$.*

Remark 6.4. We make a couple of further observations:

1. We have that $\mathrm{WF}(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(X)$. This follows directly from the definitions;
2. Given two pseudodifferential operators A and B, we have that

$$\mathrm{WF}(A \circ B) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B). \tag{6.1}$$

This is because at least one of σ_A and σ_B is of class $S^{-\infty}$ on $\mathrm{WF}(A)^c \cup \mathrm{WF}(B)^c$, so $\sigma_{A \circ B}$ is also of class $S^{-\infty}$ by the composition formula from Theorem 4.4, Item 2.

Let us now prove that the wavefront set of distributions behaves well under the action of PDOs.

Lemma 6.5. *If $u \in \mathcal{D}'(X)$ and $A \in \Psi^m(X)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(A) \cap \mathrm{WF}(u).$$

Proof. Let $(x_0, \xi_0) \in \big(\mathrm{WF}(A)^c \cup \mathrm{WF}(u)^c\big) \setminus 0$. Consider two cases:

Case 1: $(x_0, \xi_0) \notin \mathrm{WF}(A)$. Then if $B \in \Psi^0(X)$ elliptic at (x_0, ξ_0) with $\mathrm{WF}(B) \cap \mathrm{WF}(A) = \emptyset$, by Remark 6.4 we have $BA \in \Psi^{-\infty}(X)$ and so $BAu \in C^\infty(X)$. By definition this means that $(x_0, \xi_0) \notin \mathrm{WF}(Au)$. (To construct B we may take $B = \mathrm{Op}(\chi(x)\psi(\xi))$, where χ is localised near x_0 and ψ is positively 0-homogeneous and non-zero near $\xi_0/|\xi_0|$.)

Case 2: $(x_0, \xi_0) \notin \mathrm{WF}(u)$. Let $B \in \Psi^m(X)$ be elliptic at (x_0, ξ_0) such that $Bu \in C^\infty(X)$. Let V be an open conic neighbourhood of (x_0, ξ_0), such that B is elliptic at every point of V. Similarly to the parametrix construction in Theorem 5.1, we can construct $c \in S^{-m}(V)$ such that

$$c \# \sigma_B \equiv 1 \mod S^{-\infty}(V). \tag{6.2}$$

Here we introduce the notation $\bullet\#\bullet$ for expressions appearing in the symbol of a composition by (see Theorem 4.4, Item 2, and note that this operation is well-defined modulo $S^{-\infty}(T^*X)$)

$$a \# b := \sum_{\alpha \geq 0} \frac{\partial_x^\alpha a \cdot D_\xi^\alpha b}{\alpha!}.$$

Let $\chi(x, \xi) \in C^\infty(T^*X)$, such that $\chi = 0$ for small $|\xi|$, χ positively 0-homogeneous for $|\xi| \geq 1$, $\chi = 1$ on a closed conic neighbourhood $W \Subset V$ of (x_0, ξ_0) and supported in a slightly larger cone. Define $C_1 := \mathrm{Op}(\chi c) \in \Psi^{-m}(X)$ and note that $D := C_1 \circ B \in \Psi^0(X)$ satisfies $\sigma_D \equiv \chi c \# \sigma_B \mod S^{-\infty}(T^*X)$ by Theorem 4.4, Item 2. Thus $Bu \in C^\infty(X)$ implies $Du \in C^\infty(X)$. On the other hand, $\sigma_D \equiv 1 \mod S^{-\infty}$ in W by (6.2) and so by definition $\mathrm{WF}(\mathrm{Id} - D) \subset W^c$.

Take $E \in \Psi^0(X)$ elliptic at (x_0, ξ_0) with $\mathrm{WF}(E) \subset W$. Then by (6.1) we get that $\mathrm{WF}\big(EA(\mathrm{Id} - D)\big) = \emptyset$ and so $EA \equiv EAD \mod \Psi^{-\infty}$. Therefore

$$EAu \equiv EA(Du) \equiv 0 \mod C^\infty(X),$$

which by definition implies $(x_0, \xi_0) \notin \mathrm{WF}(Au)$. □

The next claim rigorously relates the notions of the wavefront set and the singular support of a distribution.

Proposition 6.6. *Let $\Pi : T^*X \to X$ be the natural projection. Then $\Pi\big(\mathrm{WF}(u)\big) = \operatorname{sing\,supp}(u)$.*

Proof. We show the inclusions from both sides and separate two cases:

Step 1: $\Pi\big(\mathrm{WF}(u)\big) \subset \operatorname{sing\,supp}(u)$. Let $x_0 \in X \setminus \operatorname{sing\,supp}(u)$. By definition, there exists $\chi \in C_0^\infty(X)$ such that $\chi u \in C^\infty(X)$. Multiplication with χ is a PDO with symbol $\chi \in S^0(X)$, which is elliptic at every (x_0, ξ_0), for $\xi_0 \neq 0$. This ends the first part of the proof.

Step 2: $\operatorname{sing\,supp}(u) \subset \Pi\big(\mathrm{WF}(u)\big)$. Let $x_0 \notin \Pi\big(\mathrm{WF}(u)\big)$. By definition, for every $\xi \in \mathbb{S}^{n-1}$ there exists $A_\xi \in \Psi^0(X)$, elliptic at (x_0, ξ) such that $A_\xi u \in C^\infty(X)$. By compactness, we may find finitely many $A_1, \dots, A_N \in \Psi^0(X)$ such that $A_j u \in C^\infty(X)$ and for every $\xi \in \mathbb{S}^{n-1}$, at least one of A_j is elliptic at (x_0, ξ). Therefore

$$A := \sum_{j=1}^N A_j^* A_j \quad \in \Psi^0(X)$$

is elliptic at x_0, and $Au \in C^\infty(X)$. By Corollary 5.3, Item 2, we have u is C^∞ near x_0, completing the proof. $\square$

We next recall a fact from the theory of distributions.

Exercise 6.7. *If $u \in \mathcal{E}'(X)$, the Fourier transform $\widehat{u}(\xi)$ belongs to $C^\infty(\mathbb{R}^n)$, and is moreover analytic. If N is the order of u (i.e. the N in the estimate (1.2)), there exists $C > 0$ with* [4]

$$|\widehat{u}(\xi)| \leq C(1 + |\xi|)^N, \quad \forall \xi \in \mathbb{R}^n. \tag{6.3}$$

Proof. By definition $\widehat{u}$ is well-defined as a distribution in $\mathcal{S}'(\mathbb{R}^n)$. We claim that $\widehat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}\rangle$. Indeed, for $\phi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\langle \widehat{u}, \phi\rangle = \langle u, \widehat{\phi}\rangle = \Big\langle u(x), \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(\xi)\, d\xi\Big\rangle = \int_{\mathbb{R}^n} \langle u(x), e^{-ix\cdot\xi}\rangle \phi(\xi)\, d\xi,$$

where in the last line we wrote $\widehat{\phi}$ as a limit of Riemann sums converging in $C^\infty(\mathbb{R}^n)$ (as in the proof of Theorem 4.3, Step 4) and used that u belongs to the dual space of $C^\infty(\mathbb{R}^n)$ (by Proposition 1.14). This proves the claim.

The claim about smoothness of $\widehat{u}$ now follows from the expression $\widehat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}\rangle$ and the fact that u extends to $C^\infty(\mathbb{R}^n)$ continuously. The analyticity claim follows from

$$\partial_\xi^\alpha \widehat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}(-ix)^\alpha\rangle$$

and the bound (1.2). More precisely, we have that for each compact set $K \subset \mathbb{R}^n$, there exists $C > 0$ such that $\sup_{\xi\in K} |\partial_\xi^\alpha \widehat{u}(\xi)| \leq C^{|\alpha|+1}\alpha!$ holds.

The estimate (6.3) follows directly from (1.2) and the relation $\widehat{u}(\xi) = \langle u(x), e^{-ix\cdot\xi}\rangle$. $\square$

Finally, we give an alternative definition of the wavefront set, in terms of directions in which the Fourier transform is rapidly decaying.

Proposition 6.8. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \in T^*X \setminus 0$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if and only if there exists $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ and a conical neighbourhood Γ of ξ_0 in $\mathbb{R}^n \setminus 0$, such that for all $N > 0$ there exists $C_N > 0$ with:*

$$|\widehat{\varphi u}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \forall \xi \in \Gamma. \tag{6.4}$$

Proof. Step 1: assume that φ, u, Γ satisfy (6.4). Let $\chi \in C^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\chi) \subset \Gamma$, χ positively 0-homogeneous for $|\xi| \geq 1$ and $\chi\big(\frac{\xi_0}{|\xi_0|}\big) = 1$. Define $a(x, y, \xi) = \varphi(x)\varphi(y)\chi(\xi) \in S^0(X \times X \times \mathbb{R}^n)$ and $A = \mathrm{Op}(a) \in \Psi^0(X)$. By definition we have that

$$Au(x) = \varphi(x)\mathcal{F}^{-1}\big(\chi(\xi)\widehat{\varphi u}(\xi)\big).$$

By (6.4) $(1+|\xi|^2)^s\chi(\xi)\widehat{\varphi u}(\xi) \in L^2(\mathbb{R}^n)$ for any $s \in \mathbb{R}$ and so $\mathcal{F}^{-1}\big(\chi(\xi)\widehat{\varphi u}(\xi)\big) \in \cap_{s\in\mathbb{R}} H^s(\mathbb{R}^n)$. By Sobolev embedding theorem (Proposition 5.7, Item 7) we obtain $Au \in C^\infty(X)$, and since A is elliptic at (x_0, ξ_0) by construction, $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

Step 2: assume $(x_0, \xi_0) \notin \mathrm{WF}(u)$. By definition there is $A \in \Psi^0(X)$ elliptic at (x_0, ξ_0) with $Au \in C^\infty(X)$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ be positively 0-homogeneous for $|\xi| \geq 1$ supported in a small cone close to $\frac{\xi_0}{|\xi_0|}$ such that $\chi\big(\frac{\xi_0}{|\xi_0|}\big) \neq 0$, and $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ and

[4] For the *Fourier-Laplace transform* and more precise relations between $u \in \mathcal{E}'(\mathbb{R}^n)$ and the decay of $\widehat{u}(\xi)$ as $|\xi| \to \infty$, see the Paley-Wiener theorem [3, Chapter 10].

sufficiently small support. As in the proof of Lemma 6.5, Case 2, there exist $B \in \Psi^0(X)$ and $R \in \Psi^{-\infty}(X)$ with

$$\mathrm{Op}\big(\chi(\xi)\varphi(y)\big) = BA + R. \tag{6.5}$$

Write $v := \varphi u \in \mathcal{E}'(X)$. By (6.5) we have $\mathrm{Op}(\chi)v \in C^\infty(X)$ and in fact, we claim that $\mathrm{Op}(\chi)v \in \mathcal{S}(\mathbb{R}^n)$. To prove the claim, note that $\mathrm{Op}(\chi)\bullet = k * \bullet$ where

$$k(x) = \mathcal{F}^{-1}(\chi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\chi(\xi)\, d\xi \quad \in \mathcal{D}'(\mathbb{R}^n).$$

As in the proof of Theorem 5.10, Step 2, integrating by parts using $(-\Delta_\xi)^N$ we get that $(1-\psi)k \in \mathcal{S}(\mathbb{R}^n)$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ near zero. Thus for large $|x|$, that is, for x outside of $\operatorname{supp} v$, the integral

$$\mathrm{Op}(\chi)v(x) = \int_{\mathbb{R}^n} k(x-y)v(y)\, dy \tag{6.6}$$

converges absolutely. Since $|x| \leq 2|x-y|$ for $y \in \operatorname{supp} v$ and $|x|$ large enough, we get that $\mathrm{Op}(\chi)v(x) = \mathcal{O}_N(|x|^{-N})$ as $|x| \to \infty$ for any $N > 0$. Repeatedly differentiating (6.6), this implies that $\mathrm{Op}(\chi)v \in \mathcal{S}(\mathbb{R}^n)$ and proves the claim.

Given the claim, we have $\chi(\xi)\widehat{\varphi u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, which shows the decay condition (6.4) in a small cone around x_0. □

Example 6.9. Let us give a few explicit examples of distributions and their wavefront sets, using the characterisation of Proposition 6.8.

1. $\mathrm{WF}(\delta_0) = \{(0,\xi) \mid \xi \neq 0\}$, where $\delta_0(x) \in \mathcal{E}'(\mathbb{R}^n)$ is the Dirac delta at zero. This follows since $\widehat{\varphi\delta}(\xi) = \varphi(0)$ for $\varphi \in C_0^\infty(X)$;
2. More generally if $S \subset \mathbb{R}^n$ is a submanifold, then $\mathrm{WF}(\delta_S) = N^*S \setminus 0 = \{(x,\xi) \mid x \in S,\, 0 \neq \xi \perp T_xS\}$;
3. Let $f \in C^\infty(\mathbb{R}^{n''})$ and δ_0 be the Dirac delta at zero in $\mathbb{R}^{n''}$. If $u(x',x'') := \delta_0(x') \otimes f(x'')$, then $\mathrm{WF}(u) = \{(0,x'',\xi',0) \mid x'' \in \operatorname{supp}(f)\}$.

   Proof. Assume first that $f(x'') \neq 0$ for all x''. Then $\mathrm{WF}(\delta \otimes f) = \mathrm{WF}(\delta \otimes \mathbf{1})$, by Lemma 6.5 (where we see the multiplication by $f(x'')$ as an invertible pseudodifferential operator). Secondly, if $\psi \in C_0^\infty(\mathbb{R}^{n'+n''})$, then $\widehat{\psi\delta_0 \otimes \mathbf{1}}(\xi',\xi'') = \widehat{\psi_0}(\xi'')$, where $\psi_0(x'') = \psi(0,x'')$. If $x' \neq 0$, then $u(x',x'') = 0$; if $\xi'' \neq 0$, then since $\widehat{\psi_0}$ is of rapid decay near ξ'', we obtain $\mathrm{WF}(u) \subset \{(0,x'',\xi',0) \mid \xi' \neq 0\}$. Assume now $\psi(0,x'') \neq 0$. The other inclusion follows by observing that for $\widehat{\psi_0}(\xi'')$ to be of rapid decay near $(\xi',0)$, implies $\widehat{\psi_0}(\xi'') = 0$ for all ξ'', so $\psi_0 \equiv 0$ contradicting the assumption.

   The case when f is allowed to vanish follows similarly. □
4. $\mathrm{WF}(H(x_1) \otimes \mathbf{1}(x'')) = \{(0,x'',\xi_1,0) \mid \xi_1 \neq 0\}$. Here $H(x_1)$ is the one-dimensional *Heaviside step function*, $H(x_1) = 1$ for $x_1 \geq 0$ and zero otherwise, and $x'' \in \mathbb{R}^{n-1}$.

   Proof. Denote $F(x_1,x'') = F(x) = H(x_1) \otimes \mathbf{1}(x'')$. First observe that $\frac{\partial F}{\partial x_1} = \delta_0(x_1) \otimes \mathbf{1}(x'')$, so by Lemma 6.5 and Item 2 above, $\{(0,x'',\xi_1,0) \mid \xi_1 \neq 0\} \subset \mathrm{WF}(F)$.

   To show this is all, consider (x_1,x'',ξ_0,ξ_0'') and note if $x_1 \neq 0$, then F smooth; if $x_1 = 0$ and $\xi_0'' \neq 0$ we let $\varphi \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^{n-1})$. Then $\widehat{\varphi H \otimes \psi}(\xi_1,\xi'') = \widehat{\varphi H}(\xi_1)\widehat{\psi}(\xi'')$ and consider $C > 0$, such that the cone $V = \{|\xi_1| < C|\xi''|\}$ contains

(ξ_0, ξ_0''). Since $\widehat{\psi}(\xi'')$ is Schwartz, we see that in V, $\widehat{\varphi H}(\xi_1)\widehat{\psi}(\xi'')$ is of rapid decay, so $(0, x'', \xi_0, \xi_0'') \notin \mathrm{WF}(F)$, completing the proof. $\square$

5. Let φ be a real-valued phase function on $X \times \mathbb{R}^N \setminus 0$, $a \in S^m(X \times \mathbb{R}^N)$, and set:

$$\Lambda_\varphi = \{(x, \varphi_x'(x,\theta)) \mid \varphi_\theta'(x,\theta) = 0,\, \theta \neq 0\}$$

which is a closed conic set in $T^*X \setminus 0$. Recall the oscillatory integral $I(a,\varphi) \in \mathcal{D}'(X)$ was defined in (2.8); then:

$$\mathrm{WF}(I(a,\varphi)) \subset \Lambda_\varphi.$$

In particular, if $A \in \Psi^m(X)$ then its Schwartz kernel K_A satisfies

$$\mathrm{WF}(K_A) \subset \{(x, x, \xi, -\xi) \mid x \in X,\, \xi \neq 0\}.$$

Proof. To prove this, we first recall that the critical set C_φ is given by $C_\varphi = \{(x,\theta) \in X \times \mathbb{R}^N \setminus 0 \mid \varphi_\theta'(x,\theta) = 0\}$ (see (2.14)). By Proposition 2.15, Item 2, we may assume that a is supported in a small closed conic neighbourhood Γ of C_φ without affecting the wavefront set and also that $a(x,\theta) = 0$ for $|\theta|$ small enough.

Let $(x_0, \xi_0) \in T^*X \setminus \big(C_\varphi \cup 0\big)$ and choose $\psi \in C_0^\infty(X)$ supported near x_0, such that $\xi_0 \neq \varphi_x'(x,\theta) \neq 0$ for all $(x,\theta) \in \Big(\operatorname{supp}\psi \times \mathbb{R}^N \setminus 0\Big) \cap \Gamma$ (we can do so by compactness and as φ has no critical points and is positively 0-homogeneous).

Again by positive 0-homogeneity of $x \cdot \xi$ and $\varphi(x,\theta)$ there is a $C > 0$ such that

$$\Big|\frac{\partial}{\partial x}\big(\varphi(x,\theta) - x\cdot\xi\big)\Big| \geq \frac{1}{C}(|\theta| + |\xi|) \tag{6.7}$$

for $(x,\theta) \in \Big(\operatorname{supp}\psi \times \mathbb{R}^N \setminus 0\Big) \cap \Gamma$ and for ξ in a small conic neighbourhood V of (x_0, ξ_0). Here it is important that V and Γ may be assumed to be disjoint.

Define ${}^tL = \frac{1}{|\varphi_x'(x,\theta) - \xi|^2}\sum_{j=1}^n \big(\frac{\partial\varphi}{\partial x_j} - \xi_j\big) D_{x_j}$ for $(x,\xi) \in V$ and note that ${}^tL e^{i(\varphi - x\cdot\xi)} = e^{i(\varphi - x\cdot\xi)}$. Then for any $k \in \mathbb{Z}_{\geq 0}$ we have

$$J(\xi) := \widehat{\psi I(a,\varphi)}(\xi) = \langle I(a,\varphi), \psi e^{-i\bullet\cdot\xi}\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i(\varphi(x,\theta) - x\cdot\xi)} L^k\big(a(x,\theta)\psi(x)\big)\, d\theta\, dx,$$

where we integrate by parts using L many times and use the inequality (6.7) to see that the coefficients of L are in $S^{-1}(\Gamma)$. Thus $J(\xi) = \mathcal{O}(|\xi|^{-k})$ for any $k > 0$ as $|\xi| \to \infty$ and $\xi \in V$ and so $(x_0, \xi_0) \notin \mathrm{WF}(I(a,\varphi))$, which completes the proof. $\square$

7. Manifolds

Aim. Build the theory of distributions, calculus of pseudodifferential operators, and Sobolev spaces on compact smooth manifolds. We will follow mostly [6, Chapter 14].

Let M be a compact smooth manifold. Denote by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M and by $T^*M = \cup_{x\in M} T_x^*M$ the cotangent bundle, where T_x^*M denotes the dual vector space to T_xM. Write $0 := \{(x,0) \mid x \in M\} \subset T^*M$ for the zero section.

7.1. **Function spaces on manifolds.** We start by carrying over the notion of a distribution to a manifold. First of all, let $(U_i, \varphi_i)_{i\in I}$ be a cover of M by charts, that is such that each $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ is a diffeomorphism. For each $K \subset U_i$ compact, and $N \in \mathbb{Z}_{\geq 0}$ an integer, equip $C^\infty(M)$ with the LCS topology induced by the seminorms $\|u\|_{i,K,N} := \|\varphi_{i*}(u|_{U_i})\|_{C^N(K)}$ for each such triple (i, K, N).

Exercise 7.1. *$C^\infty(M)$ is a Fréchet space.*

Similarly to Section 1.2, we give the space of compactly supported smooth functions $C_0^\infty(M)$ an LCS topology. Then the space of distributions $\mathcal{D}'(M)$ on M is defined as the continuous dual to $C_0^\infty(M)$. Note that $C^\infty(M) \subset \mathcal{D}'(M)$ but this *depends on a choice of a smooth measure*[5] dx on M, so that

$$\langle u, \phi\rangle := \int_M u(x)\phi(x)\, dx, \quad \forall u \in C^\infty(M), \quad \forall \phi \in C_0^\infty(M).$$

The notion of the wavefront set carries over to $u \in \mathcal{D}'(M)$ by saying that $(x, \xi) \in T^*M \setminus 0$ if in some chart $x \in U_i$, we have

$$(\varphi_i(x), \xi \circ d\varphi_i^{-1}(\varphi_i(x))) \in \mathrm{WF}(\varphi_{i*}(u|_{U_i})) \subset \varphi_i(U_i) \times \mathbb{R}^n \setminus 0.$$

It may be checked that this well defined, i.e. that it does not depend on the choice of the chart at (x, ξ).

Let us from now on assume that M is compact.

Definition 7.2. *The following generalises L^2-Sobolev space to compact manifolds*

1. *$L^2(M) := \{u : M \to \mathbb{C}$ measurable $\mid \|u\|^2_{L^2(M)} := \int_M |u|^2\, dx < \infty\}$.*
2. *Let $s \in \mathbb{R}$ and $\{(U_i, \varphi_i)\}_{i=1}^N$ be a cover of M by coordinate charts. Then we say $u \in \mathcal{D}'(M)$ lies in $H^s(M)$, if $\varphi_{i*}(u|_{U_i}) \in H^s_{\mathrm{loc}}(\varphi_i(U_i))$ for all $i = 1, \dots, N$.*
   Given a partition of unity $\{\rho_i\}_{i=1}^N$ subordinate to U_i, define for $u \in H^s(M)$

$$\|u\|^2_{H^s(M)} := \sum_{i=1}^N \|\varphi_{i*}(\rho_i u)\|^2_{H^s(\varphi_i(U_i))}.$$

Exercise 7.3. It is possible to show that:

1. $H^s(M)$ is a Hilbert space equipped with the inner product given by polarising the above, and moreover, all norms obtained in this way are equivalent;
2. $\mathcal{D}'(M) = \cup_{s\in\mathbb{R}} H^s(M)$.

7.2. **Pseudodifferential operators on manifolds.** For a guideline on how to prove what follows, see [4, Exercise 3.4]. We do not need to assume that M is compact.

end of Lecture 10, 22.11.2022

Let us start with the definition of the space of symbols:

Definition 7.4. *The symbol space $S^m(T^*M)$ consists of smooth functions $a \in C^\infty(T^*M)$ such that for all charts (U, κ) of M we have*

$$a\big(\kappa^{-1}(x), \xi \circ d\kappa(\kappa^{-1}(x))\big) \in S^m(T^*\kappa(U)). \tag{7.1}$$

(Where $(x, \xi) \in T^*\kappa(U) = \kappa(U) \times \mathbb{R}^n$.)

[5] The definition of a distribution on a manifold that does not involve choices involves the bundle of *densities*.

It is straightforward from the definition of a symbol that if $a \in S^m(T^*X)$ where $X \subset \mathbb{R}^n$, and $\kappa : X \to Y \subset \mathbb{R}^n$ is a diffeomorphism, then the symbol defined in (7.1) belongs to $S^m(T^*Y)$. Therefore, our new definition agrees with the usual one. Again using this property, it follows that it suffices to check the symbol property on an arbitrary fixed cover of M by charts. Similarly as in the case of $\mathbb{R}^n$, we defined the subspace $S^m_{\mathrm{cl}}(T^*M) \subset S^m(T^*M)$ of *classical symbols*.

Definition 7.5. *Let $P : C^\infty(M) \to \mathcal{D}'(M)$ be a continuous operator. We say $P \in \Psi^m(M)$, the space of pseudodifferential operators on M of order m, if the following two condition hold:*

1. *There is a covering of M by charts $\{(U_i, \varphi_i)\}_{i\in\mathcal{I}}$ and $P_i \in \Psi^m(\varphi_i(U_i))$ such that for all $i \in \mathcal{I}$ and all $u \in C_0^\infty(U_i)$*
$$Pu \circ \varphi_i^{-1} = P_i(u \circ \varphi_i^{-1}). \tag{7.2}$$
2. *For every $\chi_1, \chi_2 \in C^\infty(M)$ with $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$, we have $\chi_1 P \chi_2$ extending to a continuous map*
$$\chi_1 P \chi_2 : \mathcal{D}'(M) \to C^\infty(M).$$

Define the set of *smoothing operators* as $\Psi^{-\infty}(M) := \cap_{s\in\mathbb{R}}\Psi^s(M)$.

Exercise 7.6. The following points hold:

1. It can be straightforwardly checked from the definition that $P : C_0^\infty(M) \to C^\infty(M)$;
2. We may define a suitable principal symbol map, $\sigma : \Psi^m(M) \to S^m(T^*M)/S^{m-1}(T^*M)$. For $P \in \Psi^m(M)$, a cover $(U_i, \varphi_i)_{i\in\mathcal{I}}$ by charts such that P is over U_i equal to $P_i = \operatorname{Op}(p_i)$ up to smoothing (in the sense of (7.2)), define
$$\sigma(P)(x,\xi) := \sum_i \psi_i(x) p_i\big(\varphi_i(x), \xi \circ d\psi_i^{-1}(\psi_i(x))\big).$$
where $(\psi_i)_{i\in\mathcal{I}}$ is a partition of unity subordinate to the cover. Exercise: use §4.2 to show that this is independent of any choices; if $P \in \Psi^m_{\mathrm{cl}}(M)$, then $\sigma(P) \in S^m_{\mathrm{cl}}(T^*M)$ is a well defined positively homogeneous function of order m in ξ;
3. An operator $P \in \Psi^m(M)$ is said to be *elliptic* if its local representatives P_i in (7.2) are elliptic. Equivalently, P is elliptic if any its principal symbol representatives $\sigma(P) \in S^m(T^*M)$ has a suitable asymptotic growth in the ξ variable.

Theorem 7.7. *The following holds true for PDOs on manifolds:*

1. *There is a (non-canonical) quantisation map $\operatorname{Op} : S^m(T^*M) \to \Psi^m(M)$, such that ι is injective, σ is surjective with $\sigma \circ \operatorname{Op}(\bullet) = \bullet$, and $\sigma \circ \iota = 0$:*
$$0 \to \Psi^{m-1}(M) \xhookrightarrow{\iota} \Psi^m(M) \xrightarrow{\sigma} S^m(T^*M)/S^{m-1}(T^*M) \to 0. \tag{7.3}$$
(The first map is inclusion, while the second is the principal symbol map.)
2. *Assume $A \in \Psi^{m_1}(M)$ and $B \in \Psi^{m_2}(M)$ with at least one of A and B properly supported. Then $A \circ B \in \Psi^{m_1+m_2}(M)$ and $\sigma(AB) = \sigma(A)\sigma(B)$.*
3. *Assume M is compact. Given $A \in \Psi^m(M)$, we have for all $s \in \mathbb{R}$ that $A : H^s(M) \to H^{s-m}(M)$. If $A \in \Psi^{-\infty}(M)$, then $A : H^{s_1}(M) \to H^{s_2}(M)$ is compact for any $s_1, s_2 \in \mathbb{R}$.*

Exercise 7.8. Given $A \in \Psi^m(M)$ elliptic, from this theorem we are able to deduce the existence of a parametrix $B \in \Psi^{-m}(M)$ with $AB - \mathrm{Id} = K_1 \in \Psi^{-\infty}(M)$ and $BA - \mathrm{Id} = K_2 \in \Psi^{-\infty}(M)$ (c.f. the iterative procedure in Theorem 5.1).

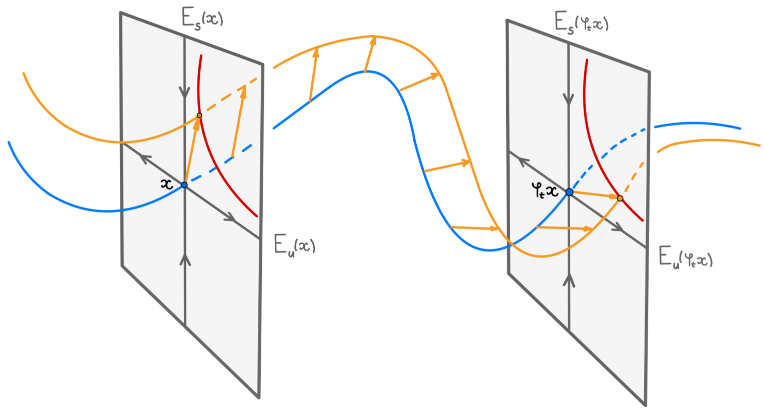

FIGURE 1. Anosov dynamics: contraction and expansion.

8. HYPERBOLIC DYNAMICS

Aim. We introduce the class of *Anosov* (or *uniformly hyperbolic*) dynamical systems, modelling chaotic dynamics. We give the main examples and construct a functional setting to studying their statistical properties.

Throughout this section $\mathcal{M}$ will be a smooth closed manifold.

8.1. **Definitions.** We introduce the basic notions of chaotic dynamics.

Definition 8.1. *We say that the flow $(\varphi_t)_{t\in\mathbb{R}}$ generated by a smooth vector field X is* Anosov *if there exists a continuous flow-invariant splitting $T\mathcal{M} = \mathbb{R}X \oplus E_u \oplus E_s$ into flow,* unstable, *and* stable directions, *respectively, and there are $C, \nu > 0$ such that*

$$|d\varphi_t(x)v| \leq Ce^{-\nu|t|} \cdot |v|, \quad \begin{cases} t \geq 0, & v \in E_s(x), \\ t \leq 0, & v \in E_u(x), \end{cases} \tag{8.1}$$

where $|\bullet|$ is an arbitrary norm on the fibres of $T\mathcal{M}$.

Example 8.2. We have the following examples of Anosov flows:

1. Let M be a closed smooth manifold and $f : M \to M$ an *Anosov diffeomorphism*, that is, we have a continuous flow-invariant splitting $TM = E_u \oplus E_s$ with the properties as in (8.1). Let $\mathcal{M} := M \times [0, 1]/\sim$ be the mapping torus associated to f, where $(x, 1) \sim (f(x), 0)$. The *suspension flow* is then defined as vertical translation $\varphi_t(x, s) = (x, s + t)$, where the addition is understood to respect $\sim$ (once we cross the value 1 we "jump" using f to the zero level). It is an Anosov flow with the splitting $T\mathcal{M} = \mathbb{R}\partial_t \oplus \widetilde{E}_u \oplus \widetilde{E}_s$, where $\widetilde{E}_{u/s}$ are the lifts of $E_{u/s}$ to $\mathcal{M}$ and ∂_t is the flow generator (exercise).
2. Let $A \in \mathrm{SL}_N(\mathbb{Z})$ be an $N \times N$ matrix with integer entries and determinant equal to 1, seen as a map on the N-torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. Say that the matrix A is *hyperbolic* if it has no eigenvalues of modulus 1. If A is hyperbolic, then $A : \mathbb{T}^N \to \mathbb{T}^N$ is Anosov with $T\mathbb{T}^N = E_u \oplus E_s$, where E_u and E_s are the span of generalised eigenvectors corresponding to eigenvalues of modulus bigger and smaller than 1, respectively (exercise).

FIGURE 2. Arnold's cat map.

3. *Arnold's cat map* is the hyperbolic matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It is one of the first examples of hyperbolic dynamics, see Figure 2.
4. Let (M, g) be a closed smooth Riemannian manifold of *negative sectional curvature*. Let $\mathcal{M} := SM := \{(x, v) \in TM \mid |v|_{g_x} = 1\}$ be the *unit sphere bundle*. Define the *geodesic flow* $\varphi_t : \mathcal{M} \to \mathcal{M}$ by $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$, where $\gamma_{x,v}(t)$ is the unit speed *geodesic*, defined by initial conditions
$$\gamma_{x,v}(0) = x, \qquad \dot{\gamma}_{x,v}(0) = v.$$
Then, the geodesic flow is Anosov (to discuss the proof for surfaces).

Dually to the differential $d\varphi_t(x)$ acting on $T_x\mathcal{M}$ there is the transpose $d\varphi_t^\intercal(x)$ acting on $T_x^*\mathcal{M}$ by $d\varphi_t^\intercal(x)\xi = \xi \circ d\varphi_t^{-1}(x) \in T^*_{\varphi_t x}\mathcal{M}$. Let X be the generator of φ_t and define $E^*_{u/s} \subset T^*\mathcal{M}$ to be the vector bundle obtained by annihilating $E_{u/s} \oplus \mathbb{R}X$ and E_0^* as the annihilator of $E_u \oplus E_s$. Let $(\Phi_t)_{t\in\mathbb{R}}$ be the flow extension $\Phi_t(x, \xi) := (\varphi_t x, d\varphi_t^\intercal(x)\xi)$ of φ_t to $T^*\mathcal{M}$ and denote by $\mathbb{X}$ its generating vector field.

We see that the flow Φ_t leaves the vector bundles $E^*_{u/s/0}$ invariant, that there is a splitting
$$T^*\mathcal{M} = E_0^* \oplus E_u^* \oplus E_s^*$$
such that with the same $C, \nu > 0$ as in Definition 8.1:
$$|d\varphi_t^\intercal(x)\xi| \leq Ce^{-\nu|t|}|\xi|, \qquad \begin{cases} t \geq 0, & \xi \in E_s^*(x), \\ t \leq 0, & \xi \in E_u^*(x). \end{cases} \tag{8.2}$$

Let us fix an arbitrary Riemannian metric g on $\mathcal{M}$ inducing a norm $|\bullet|$ on the fibres of $T^*\mathcal{M}$. Define $S^*\mathcal{M} := \{(x, \xi) \mid |\xi| = 1\}$ the unit co-sphere bundle (alternatively, we may consider the space of spherification, i.e. the space of half-lines), and denote by $\pi : T^*\mathcal{M} \setminus 0 \to S^*\mathcal{M}$ the projection. Note that Φ_t descends to $S^*\mathcal{M}$ by asking $\widetilde{\Phi}_t(x, \xi) := \left(\varphi_t(x), \frac{d\varphi_t^\intercal(x)\xi}{|d\varphi_t^\intercal(x)\xi|}\right)$. Write $\widetilde{\mathbb{X}}$ for the vector field generating $\widetilde{\Phi}_t$, and $H(x, \xi) := \xi(X(x))$ (note that Φ_t is the *Hamiltonian flow* of H with respect to the canonical symplectic structure on $T^*\mathcal{M}$). Then it follows from (8.2) that the flow $\widetilde{\Phi}_t$ has the following *sink and source structure* on $S^*\mathcal{M}$.

8.2. **Construction of the escape function.** We will follow the construction of Faure-Sjöstrand [2]. The idea is to construct a Sobolev space that has negative regularity in a cone near E_u^*, positive in the complement of a slightly bigger neighbourhood of E_u^*, and satisfies a certain dynamical invariance property.

We start with an elementary lemma. Let us denote by $d(x, S)$ the distance (with respect to an arbitrary fixed Riemannian metric) from a point x and a set S.

Lemma 8.3. *Let $(\psi_t)_{t\in\mathbb{R}}$ be a flow on a compact manifold M generated by a vector field V and let $\Sigma_{u/s} \subset M$ be compact flow-invariant disjoint subsets such that*

$$d(\psi_t(x), \Sigma_s) \to_{t\to\infty} 0, \quad x \in M \setminus \Sigma_u, \qquad d(\psi_t(x), \Sigma_u) \to_{t\to-\infty} 0, \quad x \in M \setminus \Sigma_s.$$

Let $\varepsilon > 0$. Then there exist open neighbourhoods $W_{u/s}$ of $\Sigma_{u/s}$, $\eta > 0$, and $m \in C^\infty(M; [0, 1])$ such that:

1. *$Vm \geq 0$ on M and $Vm \geq \eta > 0$ on $M \setminus (W_u \cup W_s)$;*
2. *$m = 1$ near Σ_s and $m > 1 - \varepsilon$ on W_s;*
3. *$m = 0$ near Σ_u and $m < \varepsilon$ on W_u.*

---

end of Lecture 11, 29.11.2022

Proof. Let $T > 0$ and let $V_{u/s}$ be arbitrary open neighbourhoods of $\Sigma_{u/s}$ with disjoint closures. Set

$$W_s := M \setminus \psi_T(V_u), \qquad W_u := M \setminus \psi_{-T}(V_s).$$

By the assumption when T is large enough one has $W_{u/s} \subset V_{u/s}$, $\psi_{-t}W_u \subset V_u$ and $\psi_t W_s \subset V_s$ for $t \geq 0$. Let $m_0 \in C^\infty(M; [0, 1])$ be an arbitrary smooth function such that $m_0 = 0$ on V_u and $m_0 = 1$ on V_s. Set

$$m := \frac{1}{2T} \int_{-T}^{T} \psi_t^* m_0 \, dt. \tag{8.3}$$

Then by the Fundamental Theorem of Calculus

$$Vm(x) = \frac{1}{2T}\big(m_0(\psi_T x) - m_0(\psi_{-T} x)\big). \tag{8.4}$$

Also, by compactness and by the assumption, there exists a $T_0 > 0$ such that $x \in M \setminus (V_u \cup V_s)$ implies $\psi_{T_0} x \in V_s$ and $\psi_{-T_0} x \in V_u$. We assume $T > T_0$ and split the analysis according to the location of x:

1. If $x \in M \setminus (W_u \cup W_s) = \psi_T(V_u) \cap \psi_{-T}(V_s)$, by (8.4) then $Vm(x) = \frac{1}{2T} > 0$;
2. If $x \in W_u$, then $m_0(\psi_t x) = 0$ for $t \leq T - T_0$ and so
$$m(x) = \frac{1}{2T}\Big(\int_{-T}^{T-T_0} \underbrace{m_0(\psi_t x)}_{=0} \, dt + \int_{T-T_0}^{T} \underbrace{m_0(\psi_t x)}_{\leq 1} \, dt\Big) \leq \frac{T_0}{2T} < \varepsilon,$$
for T large enough. Since $m_0(\psi_{-t} x) = 0$ for $t \geq 0$, by (8.4) $Vm(x) \geq 0$;
3. If $x \in W_s$, then $m_0(\psi_t x) = 1$ for $t \geq -T + T_0$ and so
$$m(x) = \frac{1}{2T}\Big(\int_{-T}^{-T+T_0} \underbrace{m_0(\psi_t x)}_{\geq 0} \, dt + \int_{-T+T_0}^{T} \underbrace{m_0(\psi_t x)}_{=1} \, dt\Big) \geq \frac{2T - T_0}{2T} > 1 - \varepsilon,$$
for T large enough and similarly $Vm(x) \geq 0$.

□

Next, we need to "adapt" the Riemannian metric to the dynamics.

Lemma 8.4. *Let $\nu > 0$ be the constant in (8.1). For any $\mu \in (0, \nu)$, there are smooth Riemannian metrics $g_{s/u}$ on $\mathcal{M}$ such that in (8.2) we may take $C = 1$ for the action of $d\varphi_t$ on $E_{s/u}$ with respect to metrics $g_{s/u}$. More precisely for any $x \in \mathcal{M}$ and $t \geq 0$:*

$$|d\varphi_t(x)v|_s^2 \leq e^{-\mu t}|v|_s^2,\, v \in E_s(x), \qquad |d\varphi_{-t}(x)v|_s^2 \leq e^{-\mu t}|v|_u^2,\, v \in E_u(x),$$

where $|\bullet|_{s/u}^2 := g_{s/u}(\bullet, \bullet)$ are the norms induced by $g_{s/u}$.

Proof. Let us construct g_s; the case of g_u follows by considering the reversed flow. For some $T > 0$ to be specified later we set

$$|v|_s^2 := \int_0^T e^{2\mu t}|d\varphi_t(x)v|^2\, dt, \quad x \in \mathcal{M}, v \in T_x\mathcal{M}.$$

Then we compute:

$$\begin{aligned}
|d\varphi_t(x)v|_s^2 &= e^{-2\mu t}\int_0^T e^{2\mu(r+t)}|d\varphi_{r+t}(x)v|^2\, dr = e^{-2\mu t}\int_t^{t+T} e^{2\mu q}|d\varphi_q(x)v|^2\, dq \\
&= e^{-2\mu t}\Big(|v|_s^2 + \int_T^{T+t} e^{2\mu q}|d\varphi_q(x)v|^2\, dq - \int_0^t e^{2\mu q}|d\varphi_q(x)v|^2\, dq\Big) \\
&= e^{-2\mu t}|v|_s^2 + e^{-2\mu t}\int_0^t e^{2\mu q}\big(e^{2\mu T}|d\varphi_{T+q}(x)v|^2 - |d\varphi_q(x)v|^2\big)\, dq \\
&\leq e^{-2\mu t}|v|_s^2 + e^{-2\mu t}\int_0^t e^{2\mu q}\big(C^2 e^{2\mu T}e^{-2\nu T} - 1\big)|d\varphi_q(x)v|^2\, dq \leq e^{-2\mu t}|v|_s^2,
\end{aligned}$$

if T is taken sufficiently large. Here in the second and fourth equalities we simply changed variables, while in the second to last estimate we use the Anosov condition (8.1). This completes the proof. □

We use the adapted metrics to get the following:

Lemma 8.5. *Let $\nu > 0$ be the constant in (8.1). For any $\mu \in (0, \nu)$, there exists a smooth Riemannian metric g_{ad} on $\mathcal{M}$, and conical neighbourhoods $N_{s/u}$ of $E_{s/u}^*$, such that for $\xi \neq 0$*

$$\frac{\xi}{|\xi|} \in N_s \implies \mathbb{X}\log\langle\xi\rangle_{\mathrm{ad}} \leq -\mu \cdot \frac{|\xi|_{\mathrm{ad}}^2}{1+|\xi|_{\mathrm{ad}}^2}, \quad \frac{\xi}{|\xi|} \in N_u \implies \mathbb{X}\log\langle\xi\rangle_{\mathrm{ad}} \geq \mu \cdot \frac{|\xi|_{\mathrm{ad}}^2}{1+|\xi|_{\mathrm{ad}}^2}.$$

Here $\langle\bullet\rangle_{\mathrm{ad}}$ and $|\bullet|_{\mathrm{ad}}^2$ are the Japanese bracket and the norm induced by g_{ad}, respectively.

Proof. Consider the (continuous) adapted metric

$$|v|_1^2 := |v_s|_s^2 + |v_c|^2 + |v_u|_u^2, \quad x \in \mathcal{M}, v \in T_x\mathcal{M},$$

where $v_{s/u}$ denote the projections onto $E_{s/u}$ and v_c the projection onto $\mathbb{R}X$, and $|\bullet|_{s/u}^2$ were constructed in Lemma 8.4. We note that the metric is smooth along the flow, in the sense that the map $t \mapsto |d\varphi_t(x)v|_1^2$ is smooth in t by the invariance of the Anosov splitting and smoothness of $|\bullet|_{s/u}^2$.

Next, note that we have

$$\begin{aligned}\mathbb{X}(|\xi|_1^2)(x,\xi) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(|d\varphi_\varepsilon^\intercal(x)\xi|_s^2 - |\xi|_s^2) \leq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(e^{-2\mu\varepsilon}-1)|\xi|_s^2 = -2\mu|\xi|_1^2, \quad \xi\in E_s^*(x),\\ \mathbb{X}(|\xi|_1^2)(x,\xi) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(|d\varphi_\varepsilon^\intercal(x)\xi|_u^2 - |\xi|_u^2) \geq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(e^{2\mu\varepsilon}-1)|\xi|_u^2 = 2\mu|\xi|_1^2, \quad \xi\in E_u^*(x).\end{aligned} \tag{8.5}$$

For any $\delta > 0$, approximating g_1 by smooth Riemannian metrics, we may assume that the conditions (8.5) are satisfied with μ replaced by $\mu-\delta > 0$ and the smooth norm $|\bullet|_{\mathrm{ad}}^2$. Note that $\mathbb{X}(|\xi|_{\mathrm{ad}}^2)$ are 2-homogeneous in ξ, so the previous conditions also hold on conical neighbourhood $N_{s/u}$ of $E_{s/u}^*$ with $\mu-\delta$ replaced by $\mu-2\delta$. Then it is easy to conclude that for $\xi\in N_s(x)$

$$\mathbb{X}\log\langle\xi\rangle_{\mathrm{ad}} = \frac{1}{2}\frac{\mathbb{X}|\xi|_{\mathrm{ad}}^2}{1+|\xi|_{\mathrm{ad}}^2} \leq -(\mu-2\delta)\cdot\frac{|\xi|_{\mathrm{ad}}^2}{|\xi|_{\mathrm{ad}}^2}.$$

By reversing the flow we get the analogous claim in the case of the unstable bundle and N_u. Since $\mu > 0$ and $\delta > 0$ are arbitrary, the claim follows. □

From now we work with the smooth Riemannian metric g_{ad} constructed in Lemma 8.5. The following is the main result of this paragraph:

Lemma 8.6. *Let $u, n_0, s\in\mathbb{R}$ with $u<0<s$ and $u<n_0<s$. Let N_0 be an arbitrarily small conical neighbourhood of E_0^* in $T^*\mathcal{M}\setminus 0$. There exists an* order function $m\in C^\infty(T^*\mathcal{M};[u,s])$ *and an* escape function $G_m\in C^\infty(T^*\mathcal{M})$

$$G_m(x,\xi) := m(x,\xi)\log\sqrt{1+f(x,\xi)^2}, \tag{8.6}$$

where $f\in C^\infty(T^*\mathcal{M})$ satisfies for $|\xi|\geq 1$: f is positively 1-homogeneous, $f(x,\xi)=|\xi|$ in small conical neighbourhoods of E_u^* and E_s^*, and $f(x,\xi)=|H(x,\xi)|$ in a conical neighbourhood of E_0^*. Moreover:

1. *For $|\xi|\geq 1$: $m(x,\xi)$ is positively 0-homogeneous, takes values u, n_0, and s, in small conical neighbourhoods of E_u^*, E_0^*, and E_s^*, respectively, and $\mathbb{X}m\leq 0$;*
2. *G_m decreases strictly and uniformly along Φ_t, except in a conical neighbourhood N_0 of E_0^* and for small $|\xi|$, that is, there exists $R>0$ such that*
$$(x,\xi)\in T^*\mathcal{M}\setminus N_0,\ |\xi|\geq R \implies \mathbb{X}G_m(x,\xi) < -c\min(|u|,s),$$
where $c>0$ is independent of u, n_0, and s. Moreover,
$$(x,\xi)\in T^*\mathcal{M},\ |\xi|\geq R \implies \mathbb{X}G_m(x,\xi)\leq 0.$$

Proof. Step 1: construction of $m(x,\xi)$. Let $\varepsilon>0$ and denote by $\pi: T^*\mathcal{M}\setminus 0\to S^*\mathcal{M}$ the projection. Apply Lemma 8.3 to $M=S^*\mathcal{M}$ and $\psi_t=\widetilde{\Phi}_t$ with

1. $\Sigma_u=\pi(E_s^*\setminus 0)$ and $\Sigma_s=\pi(E_u^*\oplus E_0^*\setminus 0)$. We obtain $m_1\in C^\infty(T^*\mathcal{M})$ and neighbourhoods $W_{u/s}^1$ of $\Sigma_{u/s}$ such that m_1 satisfies Items 1–3 of Lemma 8.3;
2. $\Sigma_u=\pi(E_s^*\oplus E_0^*\setminus 0)$ and $\Sigma_s=\pi(E_u^*\setminus 0)$. We obtain $m_2\in C^\infty(T^*\mathcal{M})$ and neighbourhoods $W_{u/s}^2$ of $\Sigma_{u/s}$ such that m_2 satisfies Items 1–3 of Lemma 8.3.

Set $\widetilde{m} := s+(n_0-s)m_1+(u-n_0)m_2$ and

$$\widetilde{N}_s = W_u^1\cap W_u^2,\quad \widetilde{N}_0 := W_s^1\cap W_u^2,\quad \widetilde{N}_u := W_s^1\cap W_s^2.$$

Let us split to cases according to the position of $(x,\xi)\in S^*\mathcal{M}$:

1. On $S^*\mathcal{M} \setminus (\widetilde{N}_s \cup \widetilde{N}_0 \cup \widetilde{N}_u)$ we have $\widetilde{\mathbb{X}}m_1 > \eta$ or $\widetilde{\mathbb{X}}m_2 > \eta$, and so
$$\widetilde{\mathbb{X}}\widetilde{m} = (n_0 - s)\widetilde{\mathbb{X}}m_1 + (u - n_0)\widetilde{\mathbb{X}}m_2 < -\eta \min\big(|n_0 - s|, |u - n_0|\big). \tag{8.7}$$
2. On $\widetilde{N}_s$ we have if ε is chosen small enough:
$$\widetilde{m} > s + (n_0 - s)\varepsilon + (u - n_0)\varepsilon = s(1-\varepsilon) + u\varepsilon > \frac{s}{2}. \tag{8.8}$$
3. On $\widetilde{N}_u$ we have if ε is chosen small enough:
$$\widetilde{m} < s + (n_0 - s)(1-\varepsilon) + (u - n_0)(1-\varepsilon) = \varepsilon s + u(1-\varepsilon) < \frac{u}{2}. \tag{8.9}$$
4. On the whole of $S^*\mathcal{M}$ we have:
$$\widetilde{\mathbb{X}}\widetilde{m} = (n_0 - s)\widetilde{\mathbb{X}}m_1 + (u - n_0)\widetilde{\mathbb{X}}m_2 \leq 0. \tag{8.10}$$

Finally, let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\chi(r) = 0$ for $|r| > 1$. Then, we set $m(x,\xi) := (1 - \chi(|\xi|))\widetilde{m}\Big(\frac{\xi}{|\xi|}\Big)$, a smooth function on $T^*\mathcal{M}$.

Step 2: construction of $G_m(x,\xi)$. Choose $f \in C^\infty(T^*\mathcal{M})$ such that for $|\xi| \geq 1$: f is positively 1-homogeneous, for $\frac{\xi}{|\xi|} \in \widetilde{N}_u \cup \widetilde{N}_s$ and $\frac{\xi}{|\xi|} \in \widetilde{N}_0$ we have $f(x,\xi) = |\xi|$ and $f(x,\xi) = |\langle \xi, X(x)\rangle|$, respectively. The escape function G_m is then defined by (8.6) and we have
$$\mathbb{X}G_m(x,\xi) = \mathbb{X}m \log\sqrt{1 + f(x,\xi)^2} + m\mathbb{X}\log\sqrt{1 + f(x,\xi)^2}. \tag{8.11}$$
Note firstly that $\mathbb{X}\log\sqrt{1+f^2} = \frac{\mathbb{X}f \cdot f}{1+f^2}$ is uniformly bounded by some constant C_1 (we use here that $\mathbb{X}f$ is positively 1-homogeneous for $|\xi| \geq 1$). Secondly, by Lemma 8.5 there exists $C > 0$ such that for $|\xi| \geq 1$
$$\frac{\xi}{|\xi|} \in \widetilde{N}_s \implies \mathbb{X}\log\langle\xi\rangle < -C, \qquad \frac{\xi}{|\xi|} \in \widetilde{N}_u \implies \mathbb{X}\log\langle\xi\rangle > C. \tag{8.12}$$

To show the required properties, we split to cases and assume $|\xi| \geq 1$:

1. $(x, \frac{\xi}{|\xi|}) \in S^*\mathcal{M} \setminus (\widetilde{N}_s \cup \widetilde{N}_0 \cup \widetilde{N}_u)$. Then applying (8.7) to (8.11) we get
$$\mathbb{X}G_m(x,\xi) < -\eta\min\big(|n_0 - s|, |u - n_0|\big)\log\sqrt{1 + f(x,\xi)^2} + \max(|u|, s)C_1 < -\frac{C}{2}\min(|u|, s),$$
where in the last inequality we used that f is positively 1-homogeneous, and we took $|\xi| \geq R$ with R large enough (the constant R might depend on u, n_0, or s). Note that we also used that $\mathbb{X}m(x,\xi) = \widetilde{\mathbb{X}}\widetilde{m}\Big(x, \frac{\xi}{|\xi|}\Big)$.
2. $(x, \frac{\xi}{|\xi|}) \in \widetilde{N}_s$. Then by (8.8), (8.10), and (8.12):
$$\mathbb{X}G_m(x,\xi) = \mathbb{X}m\log\langle\xi\rangle + m\mathbb{X}\log\langle\xi\rangle \leq -\frac{C}{2}|u|.$$
3. $(x, \frac{\xi}{|\xi|}) \in \widetilde{N}_u$. Then by (8.9), (8.10), and (8.12):
$$\mathbb{X}G_m(x,\xi) = \mathbb{X}m\log\langle\xi\rangle + m\mathbb{X}\log\langle\xi\rangle \leq -\frac{C}{2}s.$$
4. $(x, \frac{\xi}{|\xi|}) \in \widetilde{N}_0$. Then since $\mathbb{X}H \equiv 0$, applying (8.10) to (8.11) we have $\mathbb{X}G_m(x,\xi) \leq 0$.

Setting $c := -\frac{C}{2}$ completes the proof of the lemma. $\square$

end of Lecture 12, 13.12.2022

8.3. **Symbols and operators of variable order.** Before we get into the applications of Lemma 8.6, we need to introduce another class of symbols and operators. For an open set $X \subset \mathbb{R}^n$, we introduce the class $S^m_\rho(T^*X)$ of symbols satisfying bounds as in (2.2) for $(\rho, \delta) = (\rho, 1-\rho)$. In what follows we will assume $\frac{1}{2} < \rho \leq 1$. Let $\mathcal{M}$ be a smooth manifold. We say that $m(x,\xi) \in S^0(T^*X)$ with $\sup_{(x,\xi)\in T^*X} |m(x,\xi)| < \infty$ is an *order function*. We introduce:

Definition 8.7. *Let $m(x,\xi)$ be an order function. A function $a \in C^\infty(T^*X)$ belongs to the symbol class $S^{m(x,\xi)}_\rho(T^*X)$ of* variable order *if for any compact set $K \subset X$, multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, there is $C > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C(1+|\xi|)^{m(x,\xi)-\rho|\beta|+(1-\rho)|\alpha|}, \quad (x,\xi) \in K \times \mathbb{R}^n. \tag{8.13}$$

Note that we have $S^{m(x,\xi)}_\rho(T^*X) \subset S^{\sup_{x,\xi} m(x,\xi)}_\rho(T^*X)$ and that the class $a \in S^{m(x,\xi)}_\rho(T^*X)$ behaves well under change of coordinates, that is:

Exercise 8.8. *If $\kappa : X \to Y$ is a diffeomorphism then $a(\kappa^{-1}(y), d\kappa^{-1}(y))$ belongs to $S^{m(\kappa^{-1}(y), d\kappa^{-1}(y))}_\rho(T^*Y)$.*

The product of symbols $a \in S^{m(x,\xi)}_\rho(T^*X)$ and $b \in S^{\ell(x,\xi)}_\rho(T^*X)$ (where ℓ is another order function) satisfies $ab \in S^{m(x,\xi)+\ell(x,\xi)}_\rho(T^*X)$ similarly to Proposition 2.6. Notice that if $\rho_1 > \rho_2$ then $S^{m(x,\xi)}_{\rho_1}(T^*X) \subset S^{m(x,\xi)}_{\rho_2}(T^*X)$. The definition of a symbol of variable order can naturally be carried over to manifolds as in Section 7. We will simply write $S^{m(x,\xi)}(T^*\mathcal{M}) := S^{m(x,\xi)}_1(T^*\mathcal{M})$ if $\rho = 1$ and $S^{m(x,\xi)}_{\rho-}(T^*\mathcal{M}) := \cap_{\varepsilon>0} S^m_{\rho-\varepsilon}(T^*\mathcal{M})$.

Let us introduce the notation $\langle \xi \rangle := \sqrt{1+|\xi|^2}$ usually referred to as the *Japanese bracket*. The main example of a symbol of variable order is:

Lemma 8.9. *Let $m(x,\xi) \in S^0(T^*\mathcal{M})$ be an order function. Then $\langle \xi \rangle^{m(x,\xi)} \in S^{m(x,\xi)}_{1-}(T^*\mathcal{M})$. More generally, if $f(x,\xi) \in S^1(T^*\mathcal{M})$ is real-valued and positively 1-homogeneous for large $|\xi|$, then $\left(1+f(x,\xi)^2\right)^{\frac{m(x,\xi)}{2}} \in S^{m(x,\xi)}_{1-}(T^*\mathcal{M})$.*

Proof. For simplicity, we show only the first claim while the second claim follows similarly. It suffices to prove the claim in local coordinates, i.e. to assume $\mathcal{M} = X \subset \mathbb{R}^n$. We claim that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, writing $p(x,\xi) := \langle \xi \rangle^{m(x,\xi)}$ we have

$$\partial_x^\alpha \partial_\xi^\beta p(x,\xi) = q(x,\xi) p(x,\xi),$$

where for any $\varepsilon > 0$ we have $q \in S^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}(T^*X)$. Assuming the claim, this directly proves the required symbol estimate (8.13) on p and proves the lemma.

To prove the claim, we argue by induction on $|\alpha|+|\beta|$. For $\alpha = \beta = 0$ the claim is certainly true with $q \equiv 1$, and we assume by induction it is true for all $|\alpha|+|\beta| \leq N$. Take some α' and β' with $|\alpha'|+|\beta'| = N+1$. If $|\alpha'| \neq 0$, then we may write $\alpha' = \alpha + \mathbf{e}_i$ for some i and $\beta' = \beta$ with $|\alpha|+|\beta| = N$, with $\partial_x^\alpha \partial_\xi^\beta p = qp$ for some $q \in S^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}(T^*X)$, so:

$$\partial_x^{\alpha'} \partial_\xi^{\beta'} p(x,\xi) = \left(\partial_{x_i} q(x,\xi) + q(x,\xi) \log\langle \xi \rangle \left(\partial_{x_i} m(x,\xi)\right)\right) \langle \xi \rangle^{m(x,\xi)}. \tag{8.14}$$

The first term in the brackets in (8.14) belongs to $S^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}(T^*X)$. Moreover, since $\log\langle\xi\rangle \in S^\delta(T^*X)$ for any $\delta > 0$, the second term lives in $S^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|+\delta}(T^*X)$. Taking ε and $\delta > 0$ small enough proves the induction step in this case.

In the other case $|\beta'| \neq 0$ so we may write $\alpha' = \alpha$ and $\beta' = \beta + \mathbf{e}_i$ for some i, with $\partial_x^\alpha \partial_\xi^\beta p = qp$, so that:

$$\partial_x^{\alpha'} \partial_\xi^{\beta'} p(x,\xi) = \big(\partial_{\xi_i} q(x,\xi) + q(x,\xi)\partial_{\xi_i}\big(\log\langle\xi\rangle m(x,\xi)\big)\big)\langle\xi\rangle^{m(x,\xi)}. \tag{8.15}$$

We analyse similarly the terms in (8.15): the first term belongs to $S^{-1-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}(T^*X)$ by the induction hypothesis, while the second one belongs to $S^{-1-(1-\varepsilon)|\beta|+\varepsilon|\alpha|+\delta}(T^*X)$ for any $\delta > 0$. This similarly completes the proof of the induction step and the lemma follows. $\square$

As in Section 2 it is possible to quantise symbols in $S_\rho^{m(x,\xi)}(T^*X)$ to obtain pseudodifferential operators of *variable order* $\Psi_\rho^{m(x,\xi)}(X) \subset \Psi_\rho^{\sup_{x,\xi} m(x,\xi)}(X)$, and more generally to transfer this definition to manifolds and obtain classes $\Psi_\rho^{m(x,\xi)}(\mathcal{M})$. We say that $P = \mathrm{Op}(p) \in \Psi_\rho^{m(x,\xi)}(X)$ is *elliptic* if for every compact set $K \subset X$, there exists $C > 0$ such that (similarly to (5.1))

$$|p(x,\xi)| \geq \frac{1}{C}(1+|\xi|)^{m(x,\xi)}, \quad x \in K, \quad |\xi| \geq C.$$

Similarly one introduces the notion of ellipticity in $\Psi_\rho^{m(x,\xi)}(\mathcal{M})$. Note that in general this ellipticity is *not the same* as ellipticity in say $\Psi_\rho^{\sup_{x,\xi} m(x,\xi)}(\mathcal{M})$.

Exercise 8.10. *In fact, we have:*

1. *Following Proposition 4.6, it is possible to show that pseudodifferential operators in $\Psi_\rho^{m(x,\xi)}(X)$ behave well under change of coordinates, and that the principal symbol map is well-defined modulo $S_\rho^{m(x,\xi)-(2\rho-1)}(T^*X)$. (Note that it is here that we need $\rho > \frac{1}{2}$.) This similarly works on manifolds as in Section 7;*
2. *Similarly to Theorem 4.4 (Item 2), if $P \in \Psi_\rho^{m(x,\xi)}(X)$ and $Q \in \Psi_\rho^{\ell(x,\xi)}(X)$ for some order functions m and ℓ, and assuming at least one of P and Q is properly supported, then $PQ \in \Psi_\rho^{m(x,\xi)+\ell(x,\xi)}(\mathcal{M})$ and*

$$\sigma(PQ) - \sigma(P)\sigma(Q) \in S_\rho^{m(x,\xi)+\ell(x,\xi)-(2\rho-1)}(T^*X).$$

   This similarly works on manifolds;
3. *Similarly to Theorem 4.4 (Item 1), if $P \in \Psi_\rho^{m(x,\xi)}(\mathcal{M})$, then its formal adjoint P^* belongs to $\Psi_\rho^{m(x,\xi)}(\mathcal{M})$ such that*

$$\sigma(P^*) - \overline{\sigma(P)} \in S^{m(x,\xi)-(2\rho-1)}(T^*\mathcal{M});$$

4. *Similarly to Theorem 5.1, if $P \in \Psi_\rho^{m(x,\xi)}(X)$ is elliptic, then there exists a parametrix $Q \in \Psi_\rho^{-m(x,\xi)}(X)$ such that $PQ - \mathrm{Id}$, $QP - \mathrm{Id} \in \Psi^{-\infty}(X)$, which similarly works over manifolds as seen in Section 7.*
5. *Mapping properties: for $P \in \Psi_\rho^{m(x,\xi)}(X) \subset \Psi_\rho^{m^+}(X)$ where $m^+ = \sup_{x,\xi} m(x,\xi)$, following the proofs of Theorems 5.5 and 5.10, we get that $P : H^s_{\mathrm{comp}}(X) \to H^{s-m^+}_{\mathrm{loc}}(X)$ is continuous for each $s \in \mathbb{R}$; this carries over to manifolds as in Section 7.*

Let us recall at this point that a continuous linear map $A : D \to E$ between Banach spaces D and E is called *Fredholm* if there exists a continuous linear map $B : E \to D$ such that $AB - \mathrm{Id}_E : E \to E$ and $BA - \mathrm{Id}_D : D \to D$ are compact operators. It is possible to show that A is Fredholm if and only if it has a finite dimension kernel, closed range $A(D) \subset Y$, and finite dimensional *co-kernel* (that is, the quotient $Y/A(D)$). The *index* of the Fredholm operator A is defined as $\mathrm{ind}(A) = \dim\big(\ker A\big) - \dim\big(Y/A(D)\big)$. It can be shown that the index of a Fredholm operator is *locally constant*, that is, if $A' : D \to E$ is another Fredholm operator with $\|A - A'\|_{X\to Y}$ sufficiently small (where $\|T\|_{X\to Y} = \sup_{\|x\|_X=1} \|Tx\|_Y$ is the *operator norm*), then $\mathrm{ind}(A) = \mathrm{ind}(A')$. Note that for any compact operator $K : X \to Y$, $A+K$ is Fredholm, and $\mathrm{ind}(A) = \mathrm{ind}(A+K)$ since the index of Fredholm operators $A+sK$ for $s \in [0,1]$ is locally constant.

In the following we assume that $\mathcal{M}$ is closed. We recall that a pseudodifferential operator P is called *formally self-adjoint* if $\langle Pu, v\rangle_{L^2} = \langle u, Pv\rangle_{L^2}$ for all $u, v \in C^\infty(\mathcal{M})$.

Lemma 8.11. *Let $p \in S_\rho^{m(x,\xi)}(T^*\mathcal{M})$ be a real-valued, elliptic symbol. Then there is a formally self-adjoint operator $P \in \Psi_\rho^{m(x,\xi)}(T^*\mathcal{M})$ such that $P - \mathrm{Op}(p) \in \Psi_\rho^{m(x,\xi)-(2\rho-1)}(\mathcal{M})$, $P : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is invertible and its inverse P^{-1} belongs to $\Psi_\rho^{-m(x,\xi)}(\mathcal{M})$.*

Proof. First of all, write $Q := \frac{\mathrm{Op}(p)+\mathrm{Op}(p)^*}{2}$ and note that $\sigma(Q) - p \in S_\rho^{m(x,\xi)-(2\rho-1)}(T^*\mathcal{M})$ since p is real-valued (by Exercise 8.10, Item 3). Therefore, Q is also a formally self-adjoint, elliptic operator in $\Psi_\rho^{m(x,\xi)}(\mathcal{M})$ and in what flows we will produce a smoothing perturbation of Q that satisfies the conditions of the lemma.

Note that Q maps $H^s(\mathcal{M}) \to H^{s-m^+}(\mathcal{M})$, where $m^+ = \sup_{x,\xi} m(x,\xi)$ and $s \in \mathbb{R}$ is arbitrary (Exercise 8.10, Item 5). By the existence of the parametrix (Exercise 8.10, Item 4), there is a $B \in \Psi_\rho^{-m(x,\xi)}(\mathcal{M})$ such that $BQ - \mathrm{Id} = R$ is smoothing. If $Qu = 0$ with $u \in H^s(\mathcal{M})$, then $(\mathrm{Id} + R)u = 0$, which in particular implies that u is smooth. Now $\mathrm{Id} + R : H^s(\mathcal{M}) \to H^s(\mathcal{M})$ is a Fredholm operator (since for instance $R : H^s(\mathcal{M}) \to H^{s+1}(\mathcal{M})$ and $H^{s+1}(\mathcal{M})$ compactly embeds into $H^s(\mathcal{M})$ by the Rellich-Kondrachov theorem, see Proposition 5.7), and so it has a finite dimensional kernel implying the same conclusion for Q.

Let $(\mathbf{e}_i)_{i=1}^r$ be an $L^2(\mathcal{M})$-orthonormal basis of $\ker Q \subset C^\infty(\mathcal{M})$ and denote by $\Pi u := \sum_{i=1}^r \langle u, \mathbf{e}_i\rangle \mathbf{e}_i$ the orthogonal projection onto $\ker Q$ (here $\langle \bullet, \bullet\rangle$ denotes the distributional pairing of $u \in \mathcal{D}'(\mathcal{M})$ with a smooth function). Note that Π maps $\mathcal{D}'(\mathcal{M})$ to $C^\infty(\mathcal{M})$ continuously, hence is a smoothing operator which is formally self-adjoint, and we claim that $P := Q + \Pi$ solves the problem.

If $Pu = 0$ for some $u \in H^s(\mathcal{M})$, for an arbitrary s, then similarly as above $u \in C^\infty(\mathcal{M})$ and also $Q^2 u = 0$. Integrating by parts we get $0 = \langle Q^2 u, u\rangle_{L^2} = \|Qu\|_{L^2}^2$ which implies $Qu = 0$ and so $u \in \ker Q$, implying further $0 = \Pi u = u$, hence P is injective on each $H^s(\mathcal{M})$. Again by the parametrix construction there exists an elliptic $B \in \Psi_\rho^{-m(x,\xi)}(\mathcal{M})$ such that $BP = \mathrm{Id} + R_1$ and $PB = \mathrm{Id} + R_2$ where $R_1, R_2 \in \Psi_\rho^{-\infty}(\mathcal{M})$; we may assume that B is formally self-adjoint and that $R_1^* = R_2$. Moreover, by adding a smoothing operator to B similarly as above we may assume that B is injective. By the injectivity of B and P, we know that $\ker(\mathrm{Id} + R_1)|_{H^s(\mathcal{M})} = \{0\}$, and since $R_1 : H^s(\mathcal{M}) \to H^s(\mathcal{M})$ is compact, the operator $\mathrm{Id} + R_1$ is Fredholm of index zero. Hence $\mathrm{Id} + R_1 : H^s(\mathcal{M}) \to H^s(\mathcal{M})$ is an isomorphism for each $s \in \mathbb{R}$, which further implies that $\mathrm{Id} + R_1 : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is an isomorphism by Sobolev embedding theorem (Proposition 5.7, Item 7). In particular, writing $(\mathrm{Id} + R_1)^{-1} = \mathrm{Id} - R_1(\mathrm{Id} + R_1)^{-1}$ we get that $(\mathrm{Id} + R_1)^{-1} \in \Psi^0(\mathcal{M})$ and similarly $(\mathrm{Id} + R_2)^{-1} \in \Psi^0(\mathcal{M})$. Define $P^{-1} := B(\mathrm{Id} + R_1)^{-1} \in$

$\Psi_\rho^{-m(x,\xi)}(\mathcal{M})$ and note that by definition $PP^{-1} = (\mathrm{Id} + R_2)^{-1}BP = \mathrm{Id}$. Applying P^{-1} to the right of the second equality we get also $(\mathrm{Id} + R_2)^{-1}B = P^{-1}$ so also $P^{-1}P = \mathrm{Id}$. Since $P : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is continuous and injective with inverse $P^{-1} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, the final claim follows. □

8.4. **Existence of resonances.** Let us return to the setting of a closed manifold $\mathcal{M}$ and an Anosov flow $(\varphi_t)_{t\in\mathbb{R}} : \mathcal{M} \to \mathcal{M}$ generated by a smooth vector field X. Let $m(x,\xi)$ and $G_m(x,\xi)$ be the order and escape functions constructed in Lemma 8.6, respectively. Note that $e^{G_m} \in S_{1-}^{m(x,\xi)}(T^*\mathcal{M})$ by Lemma 8.9 and that it is elliptic and real-valued. By Lemma 8.11, there is a formally self-adjoint operator $A_m \in \Psi_{1-}^{m(x,\xi)}(\mathcal{M})$ satisfying $A_m - \mathrm{Op}(e^{G_m}) \in \Psi_{1-}^{m(x,\xi)-(2\rho-1)}(\mathcal{M})$ that is invertible on $C^\infty(\mathcal{M})$ and whose inverse A_m^{-1} belongs to $\Psi_{1-}^{-m(x,\xi)}$. Introduce the *anistropic Sobolev space* of variable order $m(x,\xi)$ as:

$$\mathcal{H}^{m(x,\xi)}(\mathcal{M}) := A_m^{-1}\big(L^2(\mathcal{M})\big), \quad \|u\|_{H^{m(x,\xi)}} := \|A_m u\|_{L^2}. \tag{8.16}$$

Since A_m is injective $\|\bullet\|_{H^{m(x,\xi)}}$ is a norm which comes from the natural scalar product $\langle u, v\rangle_{\mathcal{H}^{m(x,\xi)}} := \langle A_m u, A_m v\rangle_{L^2}$. We will consider the vector field X as a differential operator of order 1 on $\mathcal{M}$, which in particular is belongs to $\Psi^1(\mathcal{M})$ and acts on the space of distributions $\mathcal{D}'(\mathcal{M})$ (by duality). Since X does not leave the space $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ invariant, we introduce its *domain* of definition (here $\mathcal{D}(X)$ does depend on $m(x,\xi)$ which we drop from the notation for simplicity)

$$\mathcal{D}(X) := \{u \in \mathcal{H}^{m(x,\xi)}(\mathcal{M}) \mid Xu \in \mathcal{H}^{m(x,\xi)}(\mathcal{M})\}, \quad \|u\|^2_{\mathcal{D}(X)} := \|u\|^2_{\mathcal{H}^{m(x,\xi)}} + \|Xu\|^2_{\mathcal{H}^{m(x,\xi)}},$$

where the norm $\|\bullet\|_{\mathcal{D}(X)}$ comes from a natural scalar product $\langle \bullet, \bullet\rangle_{\mathcal{D}(X)}$. It follows directly from the definition that for any $z \in \mathbb{C}$ the following is a continuous linear map:

$$X + z : \mathcal{D}(X) \to \mathcal{H}^{m(x,\xi)}(\mathcal{M}).$$

---

end of Lecture 13, 15.12.2022

Proposition 8.12. *The spaces $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ and $\mathcal{D}(X)$ equipped with the scalar products $\langle \bullet, \bullet\rangle_{\mathcal{H}^{m(x,\xi)}}$ and $\langle \bullet, \bullet\rangle_{\mathcal{D}(X)}$, respectively, are Hilbert spaces.*

Proof. It suffices to show that the two spaces are complete with respect to the corresponding norms. Assume firstly that $(u_i)_{i=1}^\infty \subset \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is a Cauchy sequence. By definition, $(A_m u_i)_{i=1}^\infty \subset L^2(\mathcal{M})$ is then a Cauchy sequence, so by completeness of $L^2(\mathcal{M})$ there exists $v \in L^2(\mathcal{M})$ such that $A_m u_i \to_{i\to\infty} v$ in $L^2(\mathcal{M})$. Since A_m is invertible, we may write $v = A_m u$ where $u = A_m^{-1} v \in \mathcal{H}^{m(x,\xi)}(\mathcal{M})$, and so by definition $u_i \to_{i\to\infty} u$ in $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ proving that $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is complete.

For the domain $\mathcal{D}(X)$ of X, we argue similarly: let $(u_i)_{i=1}^\infty \subset \mathcal{D}(X)$ be a Cauchy sequence. Then by the first part there are $u, v \in \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ such that $u_i \to_{i\to\infty} u$ and $Xu_i \to_{i\to\infty} v$ in $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$. It follows that $Xu_i \to_{i\to\infty} Xu$ in $\mathcal{D}'(\mathcal{M})$ and by uniqueness of limits we conclude that $v = Xu$. Therefore $u_i \to_{i\to\infty} u$ in $\mathcal{D}(X)$, completing the proof. □

Let us make a few initial observations about the *resolvent* of X, that is, the inverse of $X + z$ (when it exists). Let Ω be an arbitrary smooth volume form on $\mathcal{M}$ (that is, a non-vanishing top degree differential form). The *divergence* $\mathrm{div}_\Omega(X) \in C^\infty(\mathcal{M})$ of X with respect to Ω is defined by the relation $\mathcal{L}_X \Omega = \mathrm{div}_\Omega(X)\Omega$, where $\mathcal{L}_X$ is the Lie derivative along the vector field X and we recall that $\mathcal{L}_X \Omega = \partial_t|_{t=0}\varphi_t^*\Omega$, where φ_t^* denotes the pullback of differential forms. Hence $\varphi_t^*\Omega = e^{\int_0^t \varphi_s^* \operatorname{div}_\Omega(X)\, ds}\Omega$ and for any $u \in C^\infty(\mathcal{M})$ we have

$$\|\varphi_{-t}^* u\|_{L^2}^2 = \int_{\mathcal{M}} |\varphi_{-t}^* u|^2\, \Omega = \int_{\mathcal{M}} |u|^2\, \varphi_t^*\Omega \leq e^{2C_0 t}\|u\|^2, \tag{8.17}$$

where $C_0 = \frac{1}{2}\sup_{x\in\mathcal{M}} |\operatorname{div}_\Omega X(x)|$, we changed variables in the second equality, and by density this holds for any $u \in L^2(\mathcal{M})$. Therefore for $\operatorname{Re}(z) > C_0$ and $u \in L^2(\mathcal{M})$ we have that $\int_0^\infty e^{-zt}\varphi_{-t}^* u\, dt$ converges in $L^2(\mathcal{M})$ and we get

$$\begin{aligned}(X+z)\int_0^\infty e^{-zt}\varphi_{-t}^* u\, dt &= (X+z)\lim_{T\to\infty}\int_0^T e^{-zt}\varphi_{-t}^* u\, dt\\ &= -\lim_{T\to\infty}\int_0^T \partial_t(e^{-zt}\varphi_{-t}^* u)\, dt = u - \lim_{T\to\infty} e^{-zT}\varphi_{-T}^* u = u,\end{aligned} \tag{8.18}$$

where in the second equality we commuted the limit as $T \to \infty$ and $X+z$ by the continuity of $X + z : L^2(\mathcal{M}) \to H^{-1}(\mathcal{M}) \subset \mathcal{D}'(\mathcal{M})$, integrated by parts in the third equality, and used that $e^{-zT}\varphi_{-T}^* u \to_{T\to\infty} 0$ in $L^2(\mathcal{M})$ in the last equality by (8.17). We just proved that

$$X + z : \mathcal{D}_{L^2}(X) := \{u \in L^2(\mathcal{M}) \mid Xu \in L^2(\mathcal{M})\} \to L^2(\mathcal{M}), \quad \operatorname{Re}(z) > C_0, \tag{8.19}$$

is an isomorphism with inverse $(X+z)^{-1} := \int_0^\infty e^{-zt}\varphi_{-t}^*\, dt$, with the natural norm $\|u\|_{\mathcal{D}_{L^2}(X)}^2 := \|u\|_{L^2}^2 + \|Xu\|_{L^2}^2$ on $\mathcal{D}_{L^2}(X)$ (coming from a Hilbert space structure similarly to Proposition 8.12).

Theorem 8.13. *With C_0 defined in (8.17) and u, s, c_0 in Lemma 8.6, for $\operatorname{Re}(z) > C_0 - c_0\min(|u|, s)$ we have that*

$$X + z : \mathcal{D}(X) \to \mathcal{H}^{m(x,\xi)}(\mathcal{M})$$

is a Fredholm operator of index zero. Moreover, for $\operatorname{Re}(z)$ large enough $X + z$ is invertible.

Proof. Let us divide the proof into steps. The idea is to mimic the identity in (8.18) up to finite time to construct an *approximate inverse* (inverse up to a compact operator). Set $P := A_m X A_m^{-1} \in \Psi_{1-}^1(\mathcal{M})$ and note that the claim is equivalent to showing that $P + z : \mathcal{D}_{L^2}(P) := \{u \in L^2(\mathcal{M}) \mid Pu \in L^2(\mathcal{M})\} \to L^2(\mathcal{M})$ is Fredholm, with the natural norm on $\mathcal{D}_{L^2}(P)$.

Step 1. We claim that $\varphi_{-t}^* : \mathcal{H}^{m(x,\xi)}(\mathcal{M}) \to \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is continuous for $t \geq 0$ which is by definition equivalent to showing the conjugate operator $A_m\varphi_{-t}^* A_m^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is continuous. Simply write

$$A_m\varphi_{-t}^* A_m^{-1} = \varphi_{-t}^*\varphi_t^* A_m \varphi_{-t}^* A_m^{-1}$$

and note that φ_{-t}^* is bounded on $L^2(\mathcal{M})$, the middle term $\varphi_t^* A_m\varphi_{-t}^* \in \Psi_{1-}^{m\circ\Phi_t(x,\xi)}$ by the change of variables formula (see Exercise 8.10, Item 1), and $A_m^{-1} \in \Psi_{1-}^{-m(x,\xi)}(\mathcal{M})$. By the composition formula (see Exercise 8.10, Item 2), $\varphi_t^* A_m\varphi_{-t}^* A_m^{-1}$ belongs to $\Psi_{1-}^{(m\circ\Phi_t - m)(x,\xi)}(\mathcal{M})$, so by Lemma 8.6, Item 1 (it gives that $(m\circ\Phi_t - m)(x,\xi) \leq 0$ for $|\xi| \geq 1$), we conclude that this operator belongs to $\Psi_{1-}^0(\mathcal{M})$ and so is bounded on $L^2(\mathcal{M})$, proving the claim.

Step 2. Let $T > 0$ and let $\chi_T \in C^\infty([0,\infty); [0,1])$ with compact support, such that $\chi_T(t) = 1$ for $t \leq T$ and $\chi_T(t) = 0$ for $t \geq T+1$, and $-2 \leq \chi_T'(t) \leq 0$. We observe the following identity similar to (8.18) (note however that all the integrals here are *finite*):

$$(X+z)\int_0^\infty \chi_T(t)e^{-zt}\varphi_{-t}^*\, dt = \mathrm{Id} + \int_0^\infty \chi_T'(t)e^{-zt}\varphi_{-t}^*\, dt. \tag{8.20}$$

Conjugate the previous identity with A_m to get:

$$(P+z)\underbrace{\int_0^\infty \chi_T(t)e^{-zt}A_m\varphi_{-t}^*A_m^{-1}\,dt}_{Q(z):=} = \mathrm{Id} + \underbrace{\int_0^\infty \chi_T'(t)e^{-zt}A_m\varphi_{-t}^*A_m^{-1}\,dt}_{R(z):=}. \tag{8.21}$$

Our aim is to show that $R(z)$ can be written as a sum of invertible (with small norm) and compact operators on $L^2(\mathcal{M})$. Let $b \in S^0(T^*\mathcal{M})$ be a symbol with values in $[0,1]$, positively 0-homogeneous for $|\xi| \geq 1$ such that $b = 1$ in a conical neighbourhood of $E_u^* \oplus E_s^*$ and $b = 0$ outside a slightly larger conical neighbourhood (so that in particular $b = 0$ on N_0). Set $B := \mathrm{Op}(b) \in \Psi^0(\mathcal{M})$ and write

$$K_1(z) := \int_0^\infty \chi_T'(t)e^{-zt}A_m\varphi_{-t}^*(\mathrm{Id}-B)A_m^{-1}\,dt.$$

Step 3. We claim that $K_1(z) \in \Psi^{-\infty}(\mathcal{M})$. Note that the principal symbol of X is $\sigma_X(x,\xi) = i\xi(X(x))$, which is non-zero in the complement of $E_u^* \oplus E_s^*$, and in particular X is elliptic on the set of $(x,\xi) \in T^*\mathcal{M} \setminus 0$ where $1 - b(x,\xi) \neq 0$. By the parametrix construction (see Theorem 5.1), for each $k \in \mathbb{N}$ there exists $E \in \Psi^{-k}(\mathcal{M})$ such that $X^kE = \mathrm{Id}-B+F$, where $F \in \Psi^{-\infty}(\mathcal{M})$. Therefore

$$\begin{aligned} K_1(z) + \int_0^\infty \chi_T'(t)e^{-zt}A_m\varphi_{-t}^*FA_m^{-1}\,dt &= \int_0^\infty \chi_T'(t)e^{-zt}A_m\varphi_{-t}^*X^kEA_m^{-1}\,dt \\ = \int_0^\infty \chi_T'(t)e^{-zt}(-\partial_t)^kA_m\varphi_{-t}^*EA_m^{-1}\,dt &= \int_0^\infty \partial_t^k\big(\chi_T'(t)e^{-zt}\big)A_m\varphi_{-t}^*EA_m^{-1}\,dt, \end{aligned}$$

where the term involving F belongs to $\Psi^{-\infty}(\mathcal{M})$ (using Step 1), we wrote $X^k = (-\partial_t)^k\varphi_{-t}^*$ and integrated by parts in the second and third equalities, respectively. The last term belongs to $\Psi_{1-}^{-k}(\mathcal{M})$, and since k was arbitrary, the claim follows.

Step 4. By Lemma 5.6 (its version for compact manifolds), we may write

$$G_t := \varphi_t^*A_m\varphi_{-t}^*BA_m^{-1} = D_t + K_t, \quad D_t \in \Psi_{1-}^0(\mathcal{M}), \quad K_t \in \Psi^{-\infty}(\mathcal{M}), \quad t \in [T, T+1],$$

such that $\|D_t\|_{L^2\to L^2} \leq \varepsilon + \limsup_{|\xi|\to\infty}\sup_{x\in\mathcal{M}}|\sigma_{G_t}(x,\xi)|$, for some fixed arbitrarily small $\varepsilon > 0$. Note that $\sigma_{G_t}(x,\xi) = e^{(G_m\circ\Phi_t - G_m)(x,\xi)}b(x,\xi)$ and using $b = 0$ on N_0 we get

$$|\sigma_{G_t}(x,\xi)| = e^{\int_0^t \partial_s(G_m\circ\Phi_s)(x,\xi)\,ds}b(x,\xi) = e^{\int_0^t \mathbb{X}G_m\circ\Phi_s(x,\xi)\,ds}b(x,\xi) \leq e^{-c_0\min(|u|,s)t}, \quad |\xi| \geq R. \tag{8.22}$$

Choose $\varepsilon > 0$ such that $\|D_t\|_{L^2\to L^2} \leq 2e^{-c_0\min(|u|,s)t}$ for $t \in [T, T+1]$; note that we may choose D_t and K_t to depend continuously on t. Set

$$K_2(z) := \int_0^\infty \chi_T'(t)e^{-zt}\varphi_{-t}^*K_t\,dt \in \Psi^{-\infty}(\mathcal{M}), \quad D(z) := \int_0^\infty \chi_T'(t)e^{-zt}\varphi_{-t}^*D_t\,dt,$$

where $K_2(z)$ is smoothing since $\varphi_{-t}^*K_t : \mathcal{D}'(\mathcal{M}) \to C^\infty(\mathcal{M})$ is continuous, hence smoothing. Using (8.22), properties of χ_T, and (8.17), we conclude that

$$\|D(z)\|_{L^2\to L^2} \leq 4\int_T^{T+1} e^{-\left(\mathrm{Re}(z)-C_0+c_0\min(|u|,s)\right)t}\,dt \leq \frac{4e^{-\left(\mathrm{Re}(z)-C_0+c_0\min(|u|,s)\right)T}}{\mathrm{Re}(z)-C_0+c_0\min(|u|,s)}.$$

Take T large enough so that $\|D(z)\|_{L^2\to L^2} \leq \frac{1}{2}$. Writing $K(z) := K_1(z) + K_2(z)$ we proved that

$$\mathrm{Id} + R(z) = \mathrm{Id} + D(z) + K(z),$$

where $\mathrm{Id} + D(z)$ is invertible on $L^2(\mathcal{M})$ (by Neumann series), and $K(z) \in \Psi^{-\infty}(\mathcal{M})$ is compact acting on $L^2(\mathcal{M})$, which shows that $Q(z)(\mathrm{Id} + D(z))^{-1}$ is a right approximate inverse.

Step 5. Let us show that $(\mathrm{Id} + D(z))^{-1}Q(z)$ is a left approximate inverse on $\mathcal{D}_{L^2}(P)$. It follows from (8.21) that $Q(z)$ maps $\mathcal{D}_{L^2}(P)$ to itself continuously. Observe next that P and $Q(z)$ commute and hence $[P, D(z)] = -[P, K(z)] \in \Psi^{-\infty}(\mathcal{M})$, which implies that $\mathrm{Id} + D(z)$ is a map on $\mathcal{D}_{L^2}(P)$. (Here we use that smoothing operators map $\mathcal{D}_{L^2}(P)$ to $C^\infty(\mathcal{M}) \subset \mathcal{D}_{L^2}(P)$.) Equivalently $[P, (\mathrm{Id} + D(z))] = -[P, K(z)] \in \Psi^{-\infty}(\mathcal{M})$, which implies that on $\mathcal{D}_{L^2}(P)$ we have

$$P(\mathrm{Id} + D(z))^{-1} = (\mathrm{Id} + D(z))^{-1}P + (\mathrm{Id} + D(z))^{-1}[P, K(z)](\mathrm{Id} + D(z))^{-1}.$$

This implies that $\mathrm{Id} + D(z) : \mathcal{D}_{L^2}(P) \to \mathcal{D}_{L^2}(P)$ is an isomorphism, and using that $K(z) : \mathcal{D}_{L^2}(P) \to \mathcal{D}_{L^2}(P)$ is compact (as it maps $\mathcal{D}_{L^2}(P)$ to $H^2(\mathcal{M})$, compactly contained in $H^1(\mathcal{M}) \subset \mathcal{D}_{L^2}(P)$), this completes the proof of the first part of the lemma.

Step 6. To show that $X + z : \mathcal{D}(X) \to \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is invertible for $\mathrm{Re}(z)$ large enough, it is possible (following Step 1) to show that $A_m \varphi_{-t}^* A_m^{-1}$ is a continuous family of operators on $L^2(\mathcal{M})$ for $t \geq 0$. Therefore there exists $C \geq 0$ such that $\sup_{t\in[0,1]} \|A_m \varphi_{-t}^* A_m^{-1}\|_{L^2\to L^2} = e^C$, and it follows that for arbitrary $t \geq 0$ we have

$$\|A_m \varphi_{-t}^* A_m^{-1}\|_{L^2\to L^2} \leq e^{C(t+1)}.$$

This shows that for $\mathrm{Re}(z)$ large enough, we have $\|R(z)\|_{L^2\to L^2} \leq \frac{1}{2}$ and so $\mathrm{Id} + R(z)$ is invertible on $L^2(\mathcal{M})$. By (8.18) we conclude that $X + z$ is an isomorphism for $\mathrm{Re}(z)$ large enough, completing the proof. □

Let us record an immediate corollary of this theorem to the existence of the resolvent.

Corollary 8.14. *The resolvent map* $z \mapsto (X + z)^{-1} : \mathcal{H}^{m(x,\xi)}(\mathcal{M}) \to \mathcal{D}(X) \subset \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ *is meromorphic in the region* $\mathrm{Re}(z) > C_0 - c_0 \min(|u|, s)$ *of the complex plane.*

Proof. According to Theorem 8.13, $X + z : \mathcal{D}(X) \to \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is a holomorphic family of Fredholm operators in the mentioned region (in the sense of Definition 8.17), invertible for $\mathrm{Re}(z)$ large enough. Therefore the Analytic Fredholm Theorem 8.18 applies and proves the claim directly. □

Next we show that the eigenvalues of X on $\mathcal{D}(X)$ are *intrinsic*, that is, the set of z satisfying $\mathrm{Re}(z) > C_0 - c_0 \min(|u|, s)$ for which $\ker(X + z) \neq \{0\}$ does not depend on the choice of the space $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$.

Theorem 8.15. *Let* $\widetilde{m}$ *and* $G_{\widetilde{m}}$ *be an order and an escape function respectively satisfying the properties of Lemma 8.6 for some* $\widetilde{u} < 0 < \widetilde{s}$. *Let* $D := \min(\min(|u|, s), \min(|\widetilde{u}|, \widetilde{s}))$. *Then for* z *satisfying* $\mathrm{Re}(z) > C_0 - c_0 D$, *the eigenvalues and the corresponding eigenspaces of* $X + z$ *acting on* $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ *and* $\mathcal{H}^{\widetilde{m}(x,\xi)}(\mathcal{M})$ *coincide.*

Proof. According to the second part of Theorem 8.13 and (8.19), there exists $C > 0$ sufficiently large such that $X + z$ is an isomorphism on $L^2(\mathcal{M})$, $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$, and $\mathcal{H}^{\widetilde{m}(x,\xi)}(\mathcal{M})$ (with appropriate domain) for $\mathrm{Re}(z) > C$, with inverse denoted by $R_{L^2}(z)$, $R_{\mathcal{H}^{m(x,\xi)}}(z)$, and

$R_{\mathcal{H}^{\widetilde{m}(x,\xi)}}(z)$, respectively. Let $U < 0 < S$ be such that $U < m(x,\xi), \widetilde{m}(x,\xi) < S$ for all $(x,\xi) \in T^*\mathcal{M}$; then

$$H^U(\mathcal{M}) \subset \mathcal{H}^{m(x,\xi)}(\mathcal{M}), \mathcal{H}^{\widetilde{m}(x,\xi)}(\mathcal{M}) \subset H^S(\mathcal{M}),$$

and therefore $R(z) := R_{L^2}(z) = R_{\mathcal{H}^{m(x,\xi)}}(z) = R_{\mathcal{H}^{\widetilde{m}(x,\xi)}}(z)$ on $H^S(\mathcal{M})$ for $\mathrm{Re}(z) > C$. Moreover, by Corollary 8.14 $R(z) : H^S(\mathcal{M}) \to H^U(\mathcal{M})$ admits a meromorphic continuation that agrees with $R_{\mathcal{H}^{m(x,\xi)}}(z)$ and $R_{\mathcal{H}^{\widetilde{m}(x,\xi)}}(z)$ (restricted to $H^S(\mathcal{M})$) for $\mathrm{Re}(z) > C_0 - c_0 D$. If we take z in this region and a small positively oriented circle γ around z not including any eigenvalues except possibly z, we conclude that on $H^S(\mathcal{M})$

$$\Pi_z^{\mathcal{H}^{m(x,\xi)}} = \Pi_z^{\mathcal{H}^{\widetilde{m}(x,\xi)}} = \frac{1}{2\pi i} \oint_\gamma R(w)\, dw.$$

Since $H^S(\mathcal{M})$ is dense in $\mathcal{H}^{m(x,\xi)}(\mathcal{M})$ and $\mathcal{H}^{\widetilde{m}(x,\xi)}(\mathcal{M})$, and these spectral projectors have finite rank, by Lemma 8.19 we conclude that the ranges of $\Pi_z^{\mathcal{H}^{m(x,\xi)}}$ and $\Pi_z^{\mathcal{H}^{\widetilde{m}(x,\xi)}}$ agree, which implies the required conclusion. □

Eigenvalues of $X + z$ acting on anisotropic Sobolev spaces are called *Pollicott-Ruelle resonances*. The corresponding eigenfunctions $u \in \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ are called *Pollicot-Ruelle resonant states* or *generalised* Pollicot-Ruelle resonant states if $(X + z)u = 0$ or there exists $J \geq 1$ such that $(X + z)^J u = 0$, respectively. These are well-defined by Theorem 8.15. For brevity we will often drop the 'Pollicott-Ruelle'.

Lemma 8.16. *Assume $u \in \mathcal{H}^{m(x,\xi)}(\mathcal{M})$ is a generalised resonant state satisfying $(X+z)u = 0$ for some $z \in \mathbb{C}$. Then $\mathrm{WF}(u) \subset E_u^*$ and $\mathrm{Re}(z) \leq 0$.*

Proof. Since by Theorem 8.15 the spectrum is intrinsic, we are free to choose our anisotropic Sobolev space satisfying Lemma 8.6. Indeed, we may take n_0 and s in this lemma arbitrarily large, such that $m(x,\xi)$ is arbitrarily large for $|\xi| \geq 1$ and outside of an arbitrarily small conical neighbourhood of E_u^*. It follows that $\mathrm{WF}(u) \subset E_u^*$.

For the last statement, note that the resolvent

$$(X + z)^{-1} = \int_0^\infty e^{-zt} \varphi_{-t}^*\, dt,$$

is well-defined for $\mathrm{Re}(z) > 0$ as a map $C^0(\mathcal{M}) \to C^0(\mathcal{M})$ and is invertible. Arguing similarly as in Theorem 8.15 using that $C^\infty(\mathcal{M})$ and $H^N(\mathcal{M})$ are dense in $\mathcal{H}^{m(x,\xi)}$ and $C^0(\mathcal{M})$, and that the two resolvents agree for $\mathrm{Re}(z) \gg 1$, we conclude there are no poles for $\mathrm{Re}(z) > 0$. □

8.5. **Meromorphic continuation of operators.** In this section we set up a theoretical framework for defining a meromorphic family of resolvents $z \mapsto (X + z)^{-1}$. For a reference on the material in this section, see [1, Section C.3]. Let X and Y be Banach spaces, and denote by $\mathcal{L}(X, Y)$ the Banach space of continuous linear maps between X and Y. We start with

Definition 8.17. *Let $\Omega \subset \mathbb{C}$ be an open connected set and for $z \in \Omega$ let $B(z) \in \mathcal{L}(X, Y)$. We say that $z \mapsto B(z)$ is:*

1. holomorphic*, if for every $x \in X$ and $y^* \in Y^*$ (the dual of Y), we have $\Omega \ni z \mapsto y^*(B(z)x)$ is holomorphic;*

2. meromorphic, *if for any* $z_0 \in \Omega$ *there exist operators* $(B_j)_{j=1}^J \subset \mathcal{L}(X,Y)$ *of finite rank and a family of operators* $z \mapsto B_0(z)$ *defined and holomorphic in a neighbourhood* U *of* z_0, *such that*

$$B(z) = B_0(z) + \sum_{j=1}^J \frac{B_j}{(z-z_0)^j}, \quad z \in U \setminus z_0.$$

There is a natural class of meromorphic operators arising in practise. This is recorded by the following theorem, known as the *Analytic Fredholm Theorem*:

Theorem 8.18. *Let* $\Omega \subset \mathbb{C}$ *be a connected open set and assume that* $X \subset Y$ *is a continuous inclusion. If for every* $z \in \Omega$, $P + z : X \to Y$ *is a Fredholm operator and there exists a* $z_0 \in \Omega$ *such that* $P + z_0 : X \to Y$ *is an isomorphism, then* $z \mapsto (P+z)^{-1} \in \mathcal{L}(Y,X)$ *is a meromorphic family of operators on* Ω. *Moreover, define for* $z \in \Omega$

$$\Pi_z := \frac{1}{2\pi i} \oint_z (w+P)^{-1}\, dw,$$

where the contour integral is along a small positively oriented circle around z *including no poles of* $(w+P)^{-1}$ *except possibly* z. *Then* Π_z *is a bounded projection of finite rank, that is:*

$$\Pi_z^2 = \Pi_z, \quad \Pi_z : Y \to X \subset Y,$$

and there exists a neighbourhood U *of* z, $J \in \mathbb{N}_0$, *and a holomorphic* $U \ni w \mapsto B(w) \in \mathcal{L}(Y,X)$ *such that*

$$(P+w)^{-1} = B_0(w) + \sum_{j=1}^J \frac{(-(P+z))^{j-1}\Pi_z}{(w-z)^j}, \quad w \in U \setminus z.$$

Moreover, J *is the smallest non-negative integer such that* $\ker(P+z)^J|_X = \operatorname{ran}(\Pi_z)$ *(and the composition is well-defined).*

We conclude this section with an elementary lemma.

Lemma 8.19. *Let* $F : X \to Y$ *be a finite rank operator and* $Z \subset X$ *a dense subspace. Then* $F(Z) = F(X)$.

Proof. Let $x_1, \dots, x_r \in X$ be such that $F(x_1), \dots, F(x_r)$ form a basis of $F(X)$. By the equivalence of norms on finite dimensional spaces, there is a constant $C > 0$ such that

$$\frac{1}{C} \max_i |a_i| \leq \| \sum_{i=1}^r a_i F(x_i) \|_X \leq C \max_i |a_i|, \tag{8.23}$$

for any $a_i \in \mathbb{C}$. Let $\varepsilon > 0$ and take $z_i \in Z$ such that $\|z_i - x_i\|_X < \varepsilon$ for $i = 1, \dots, r$. Then for an $r \times r$ matrix A we may write

$$\begin{pmatrix} F(z_1) \\ \vdots \\ F(z_r) \end{pmatrix} = A \begin{pmatrix} F(x_1) \\ \vdots \\ F(x_r) \end{pmatrix}$$

This implies that the norm of each entry of $(A - \mathrm{Id}) \begin{pmatrix} F(x_1) \\ \vdots \\ F(x_r) \end{pmatrix}$ is less than $\varepsilon \|F\|_{X \to Y}$, and consequently by (8.23) that each entry of $A - \mathrm{Id}$ has absolute value bounded by $C\varepsilon \|F\|_{X \to Y}$.

Taking $\varepsilon > 0$ small enough, we get that A is invertible and therefore $F(z_1), \dots, F(z_r)$ span $F(X)$, completing the proof. ☐

8.6. **Ergodicity.** In this section we assume that the Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ preserves the smooth volume form Ω. We will say that $(\varphi_t)_{t\in\mathbb{R}}$ is *ergodic* with respect to Ω if any $u \in L^1(\mathcal{M})$ satisfying $Xu = 0$ is a constant function. Note that $-X$ generates the Anosov flow $(\varphi_{-t})_{t\in\mathbb{R}}$ which has E_u and E_s as stable an unstable bundles, respectively.

Since the flow preserves Ω, we have $\|\varphi_t^* u\|_{L^p} = \|u\|_{L^p}$ for any $u \in L^p(\mathcal{M})$ and $p \in [1, \infty]$. Denote by $\mathbf{R}_\pm(z)$ the corresponding resolvents, defined on $L^p(\mathcal{M})$ for $\mathrm{Re}(z) > 0$ and any $p \in [1,\infty]$ by the following formula:

$$\mathbf{R}_\pm(z) := (\pm X + z)^{-1} = \int_0^\infty e^{-zt}\varphi_{\mp t}^*\, dt. \tag{8.24}$$

It follows that there are no Pollicott-Ruelle resonances for $\mathrm{Re}(z) > 0$ and that

$$\|\mathbf{R}_\pm(z)\|_{L^p\to L^p} \leq \frac{1}{\mathrm{Re}(z)}, \quad \mathrm{Re}(z) > 0, \quad p \in [1,\infty]. \tag{8.25}$$

Let $\mathcal{H}^\pm(\mathcal{M})$ be some anisotropic Sobolev space constructed with respect to $\pm X$ such that $\mathbf{R}_\pm(z) : \mathcal{H}^\pm(\mathcal{M}) \to \mathcal{H}^\pm(\mathcal{M})$ is meromorphic for $\mathrm{Re}(z) > -C_1$, for some $C_1 > 0$. Denote by $\Pi_\pm$ the spectral projectors at zero with respect to $\pm X$. It follows from the above formulas and meromorphic continuation that

$$\mathbf{R}_+(z)^* = \mathbf{R}_-(\overline{z}), \quad (\Pi_+)^* = \Pi_-, \tag{8.26}$$

where the superscript * denote the L^2-adjoint. We start with an auxiliary lemma.

Lemma 8.20. *Assume $u \in L^1(\mathcal{M})$ satisfies $Xu = i\lambda u$ for some $\lambda \in \mathbb{R}$. Then $u \in C^\infty(\mathcal{M})$.*

Before proving the lemma, we apply it to show ergodicity of $(\varphi_t)_{t\in\mathbb{R}}$.

Theorem 8.21. *The volume-preserving Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ is ergodic with respect to Ω.*

Proof. Let $u \in L^1(\mathcal{M})$ such that $Xu = 0$. By Lemma 8.20 we have that $u \in C^\infty(\mathcal{M})$. Let $v \in E_s(x)$ for some $x \in \mathcal{M}$. Using that $\varphi_t^* u = u$ for all $t \in \mathbb{R}$, we get $du(x)(v) = du(\varphi_t x)(d\varphi_t(x)v)$ and letting $t \to \infty$ by the Anosov condition (8.1), we get $du(x)(v) = 0$. Similarly by letting $t \to -\infty$ and $v \in E_u(x)$ we get $du(x)(v) = 0$. By the assumption $du(x)(X(x)) = 0$ and therefore $du \equiv 0$ which implies that u is constant, proving the claim. ☐

---

end of Lecture 14, 20.12.2022

Proof of Lemma 8.20. Let us assume that $\lambda = 0$ (the other cases follow similarly). Firstly, observe that constant functions are resonant states of X. Moreover, if there exists a generalised resonant state u such that $X^2u = 0$ and $Xu \neq 0$, by Lemma 8.19 we may write $u = \Pi_+ v$ for some $v \in C^\infty(\mathcal{M})$. Therefore $\mathbf{R}_+(z)v = \mathcal{O}_{\mathcal{H}^+}\big(\frac{1}{\mathrm{Re}(z)^2}\big)$ as $\mathrm{Re}(z) \to 0^+$ which contradicts (8.25). Similar argument applies to $-X$ and so we may write close to $z = 0$

$$\mathbf{R}_\pm(z) = \mathbf{R}_\pm^{\mathrm{hol}}(z) + \frac{\Pi_\pm}{z}, \tag{8.27}$$

where $z \mapsto \mathbf{R}_\pm^{\mathrm{hol}}(z) : \mathcal{H}^\pm(\mathcal{M}) \to \mathcal{H}^\pm(\mathcal{M})$ are holomorphic. Let $(u_n)_{n=1}^\infty \subset C^\infty(\mathcal{M})$ be a sequence such that $u_n \to u$ in $L^1(\mathcal{M})$, and define $v_n := \Pi_+ u_n \in \ker X|_{\mathcal{H}^+}$.

Step 1: we claim that $\|v_n\|_{\mathcal{H}^+}$ is bounded and argue by contradiction, assuming (up to taking a subsequence) $\|v_n\|_{\mathcal{H}^+} \to \infty$. Since $\ker X|_{\mathcal{H}^+}$ is finite-dimensional, we may further assume $\frac{v_n}{\|v_n\|_{\mathcal{H}^+}} \to v_\infty \in \mathcal{H}^+$ for some $v_\infty \in \mathcal{H}^+$ with $\|v_\infty\|_{\mathcal{H}^+} = 1$. Let $\psi \in C^\infty(\mathcal{M})$ be a test function. Then:

$$\begin{aligned}\langle v_\infty, \psi\rangle &= \langle v_\infty - \tfrac{v_n}{\|v_n\|_{\mathcal{H}^+}}, \psi\rangle + \tfrac{1}{\|v_n\|_{\mathcal{H}^+}}\langle u_n, \Pi_-\psi\rangle\\ &= \langle v_\infty - \tfrac{v_n}{\|v_n\|_{\mathcal{H}^+}}, \psi\rangle + \tfrac{1}{\|v_n\|_{\mathcal{H}^+}}\langle u_n, (z\mathbf{R}_-(z) - z\mathbf{R}_-^{\mathrm{hol}}(z))\psi\rangle,\end{aligned}$$

where we used (8.26) in the first equality and (8.27) in the second line. Note that

$$|\langle u_n, z\mathbf{R}_-(z)\psi\rangle| \leq \|u_n\|_{L^1}\|z\mathbf{R}_-(z)\psi\|_{L^\infty} \leq 2\|u\|_{L^1}\|\psi\|_{L^\infty}, \tag{8.28}$$

for n large enough, where we used (8.25) in the last estimate. Let us fix $\varepsilon > 0$. Then for n large enough

$$|\langle v_\infty - \tfrac{v_n}{\|v_n\|_{\mathcal{H}^+}}, \psi\rangle| \leq \varepsilon, \quad \|v_n\|_{\mathcal{H}^+}^{-1} \leq \varepsilon. \tag{8.29}$$

Note also that $z\mathbf{R}_-^{\mathrm{hol}}(z) : \mathcal{H}^+ \to \mathcal{H}^+$ is uniformly bounded, and therefore taking $|z|$ small enough (depending on n and ε), combining with (8.28) and (8.29) implies that, for some $C > 0$

$$|\langle v_\infty, \psi\rangle| \leq C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get that $v_\infty = 0$, contradicting $\|v_\infty\|_{\mathcal{H}^+} = 1$.

Step 2: now $\|v_n\|_{\mathcal{H}^+}$ is bounded, and again using that $\ker X|_{\mathcal{H}^+}$ is finite-dimensional we may assume that $v_n \to v_\infty$ in $\mathcal{H}^+$. We claim that $v_\infty = u$. For that purpose, fix $\varepsilon > 0$ and note that $(X + z)u = zu$ implies that $u = z\mathbf{R}_+(z)u$ for all z with $\mathrm{Re}(z) > 0$. Therefore for all test functions $\psi \in C^\infty(\mathcal{M})$:

$$\langle u, \psi\rangle = \langle z\mathbf{R}_+(z)(u - u_n), \psi\rangle + \langle z\mathbf{R}_+(z)u_n - v_n, \psi\rangle + \langle v_n - v_\infty, \psi\rangle + \langle v_\infty, \psi\rangle.$$

Arguing similarly to Step 1, observe that by (8.25) and for n large enough we have:

$$|\langle z\mathbf{R}_+(z)(u - u_n), \psi\rangle| \leq \|u - u_n\|_{L^1}\|\psi\|_{L^\infty} \leq \varepsilon, \quad \mathrm{Re}(z) > 0. \tag{8.30}$$

Also, since $v_n \to v_\infty$ in $\mathcal{D}'(\mathcal{M})$, taking n large enough we may assume that $|\langle v_n - v_\infty, \psi\rangle| \leq \varepsilon$. Finally, since $z\mathbf{R}_+(z)u_n - v_n = z\mathbf{R}_+^{\mathrm{hol}}(z)u_n$ by (8.27), and $\mathbf{R}_+^{\mathrm{hol}}(z) : \mathcal{H}^+ \to \mathcal{H}^+$ is uniformly bounded we observe

$$\begin{aligned}|\langle z\mathbf{R}_+(z)u_n - v_n, \psi\rangle| &= |z||\langle A_m\mathbf{R}_+^{\mathrm{hol}}(z)u_n, A_m^{-1}\psi\rangle|\\ &\leq |z|\|\mathbf{R}_+^{\mathrm{hol}}(z)\|_{\mathcal{H}^+\to\mathcal{H}^+}\|u_n\|_{\mathcal{H}^+}\|A_m^{-1}\psi\|_{L^2} \leq \varepsilon,\end{aligned}$$

for $|z|$ small enough (depending on both n and ε), where we used that $A_m^* = A_m$. Everything combined shows $|\langle u - v_\infty, \psi\rangle| \leq 3\varepsilon$, and since $\varepsilon > 0$ and ψ were arbitrary, we get $u \equiv v_\infty \in \mathcal{H}^+$.

Step 3. The same argument as in Step 2 applied to $-X$ shows that $u \in \mathcal{H}^-$. Since by Lemma 8.6 we may construct $\mathcal{H}^\pm$ such that $\mathcal{H}^+ \cap \mathcal{H}^- \subset H^{n_0}(\mathcal{M})$ and take n_0 arbitrarily large, by Sobolev embedding we get that $u \in C^\infty(\mathcal{M})$, completing the proof. □

8.7. **Mixing.** Let us now get to the stronger *mixing* properties. We say that $(\varphi_t)_{t\in\mathbb{R}}$ is mixing with respect to Ω if

$$\langle \varphi_{-t}^* f, g\rangle_{L^2} = \int_{\mathcal{M}} \varphi_{-t}^* f \cdot \bar{g}\,\Omega \to_{t\to\infty} \Big(\int_{\mathcal{M}} f\,\Omega\Big)\Big(\int_{\mathcal{M}} \bar{g}\,\Omega\Big), \qquad \forall f, g \in L^2(\mathcal{M}).$$

By subtracting from f and g their average, mixing is equivalent to the fact that $\langle \varphi_{-t}^* f, g\rangle_{L^2} \to 0$ for all $f, g \in L^2(\mathcal{M})$ with zero average. It is possible to give a spectral characterisation of mixing:

Lemma 8.22. *The volume-preserving Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ is mixing with respect to Ω if and only if there are no non-constant $L^2(\mathcal{M})$ eigenfunctions on the imaginary axis, that is, we have $u \in L^2(\mathcal{M})$ with $(X + i\lambda)u = 0$ for some $\lambda \in \mathbb{R}$ implies $\lambda = 0$ and u is constant.*

Proof. Now, if $(X + i\lambda)u = 0$ for some $0 \neq \lambda \in \mathbb{R}$ and $0 \not\equiv u \in L^2(\mathcal{M})$, then clearly u has zero average and $\varphi_{-t}^* u = e^{i\lambda t}u$ for $t \in \mathbb{R}$. Therefore $\langle \varphi_{-t}^* u, u\rangle_{L^2} = e^{i\lambda t}\|u\|_{L^2}^2$ which does not converge to zero.

It remains to show the other direction, that is, to show the flow is mixing given that there are no L^2-eigenfunctions on the imaginary axis. We will make use of the spectral theorem for unbounded self-adjoint operators. The operator $-iX$ is a closed (by Proposition 8.12), unbounded, self-adjoint operator, with domain $\mathcal{D}_{L^2}(X) \subset L^2(\mathcal{M})$. Therefore the spectral theorem applies, and by the functional calculus we have $\varphi_{-t}^* = e^{-tX}$ on $L^2(\mathcal{M})$. Given $u, v \in L^2(\mathcal{M})$ denote by $d\mu_{u,v}$ the spectral measure (defined on the Borel sets of $\mathbb{R}$) satisfying for any bounded Borel function f that

$$\langle f(-iX)u, v\rangle_{L^2} = \int_{\mathbb{R}} f \, d\mu_{u,v}.$$

Note that $d\mu_{u,v}$ is a finite measure, and when f is a characteristic function of a Borel set $f(-iX)$ is an orthogonal projector on $L^2(\mathcal{M})$. Next, we record *Stone's formula* that expresses the spectral measure in terms of the resolvent, for any real numbers $a < b$, and $u \in C^\infty(\mathcal{M})$:

$$\begin{aligned}\frac{1}{2}\big(\mathbf{1}_{[a,b]}(-iX) + \mathbf{1}_{(a,b)}(-iX)\big)u &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_a^b \big((-iX - (\lambda + i\varepsilon))^{-1} - (-iX - (\lambda - i\varepsilon))^{-1}\big)u \, d\lambda \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_a^b \big(\mathbf{R}_-(i\lambda + \varepsilon) + \mathbf{R}_+(-i\lambda + \varepsilon)\big)u \, d\lambda.\end{aligned}$$

If there are no resonances of $\pm X$ in the set $-i[a, b]$, by the meromorphic continuation we may take the limit to get $-\frac{1}{2\pi}\int_a^b \big(\mathbf{R}_-(i\lambda) + \mathbf{R}_+(-i\lambda)\big)u \, d\lambda$. Therefore

$$|d\mu_{u,v}((a, b))| \leq |b - a| \sup_{\lambda\in(a,b)} |\langle \big(\mathbf{R}_-(i\lambda) + \mathbf{R}_+(-i\lambda)\big)u, v\rangle_{L^2}|,$$

which in turn implies that $d\mu_{u,v}$ is absolutely continuous with respect to the Lebesgue measure outside of the resonances of $\pm X$. Note that this is also true for any $u, v \in L^2(\mathcal{M})$ by the continuity of the spectral measure with respect to u, v and density of $C^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$. In fact, by taking the holomorphic part of the resolvents, we get that

$$d\mu_{u,v}^{\mathrm{ac}}(\lambda) := -\frac{1}{2\pi}\big\langle \big(\mathbf{R}_-^{\mathrm{hol}}(i\lambda) + \mathbf{R}_+^{\mathrm{hol}}(-i\lambda)\big)u, v\big\rangle_{L^2} \, d\lambda$$

is the absolutely continuous part of $d\mu_{u,v}$, that is, there is a splitting

$$d\mu_{u,v} = d\mu_{u,v}^{\mathrm{ac}} + d\mu_{u,v}^{\mathrm{pp}},$$

where $d\mu_{u,v}^{\mathrm{pp}}$ is *pure point*, that is, it is supported on the resonances of $\pm X$. However, by the spectral theorem we know that $\lambda \in \operatorname{supp}(d\mu_{u,v}^{\mathrm{pp}})$ if and only if λ is an L^2-eigenvalue of X, that is, there exists $u \in L^2(\mathcal{M})$ such that $Xu = i\lambda u$. By assumption this implies that $\lambda = 0$ and u is constant. Now note that we may restrict the domain of $-iX$ to $\mathcal{D}_{L^2}(\mathcal{M}) \cap \mathbf{1}^\perp \subset L^2(\mathcal{M}) \cap \mathbf{1}^\perp$, where $\mathbf{1}^\perp$ is the L^2 orthogonal to the constant function $\mathbf{1}$ equal to 1 everywhere, that is, the space of functions with zero average. Then exactly the same argument as above applies and shows $u = 0$. Thus $d\mu_{u,v}^{\mathrm{pp}} = 0$ and $d\mu_{u,v}$ is absolutely continuous with respect to the Lebesgue measure.

Therefore by the spectral theorem we may write

$$\begin{aligned}\langle \varphi_{-t}^* u, v\rangle_{L^2} = \langle e^{-it\lambda}(-iX)u, v\rangle_{L^2} &= \int_{\mathbb{R}} e^{-it\lambda}\, d\mu_{u,v}\\ &= -\frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\lambda}\underbrace{\big\langle \big(\mathbf{R}_-^{\mathrm{hol}}(i\lambda) + \mathbf{R}_+^{\mathrm{hol}}(-i\lambda)\big)u, v\big\rangle_{L^2}}_{F(\lambda):=}\, d\lambda,\end{aligned}$$

where since $d\mu_{u,v}$ is a finite measure we get that $F(\lambda) \in L^1(\mathbb{R}, d\lambda)$. Writing $\langle \varphi_{-t}^* u, v\rangle_{L^2} = -\frac{1}{2\pi}\widehat{F}(\lambda)$ and applying the Riemann-Lebesgue lemma (see Proposition 1.19, Item 1) implies that $\langle \varphi_{-t}^* u, v\rangle_{L^2} \to_{t\to\infty} 0$.

Let us now show by a simple density argument that this implies mixing. Let $u, v \in L^2(\mathcal{M})$ with zero average and let $u_n, v_n \in C^\infty(\mathcal{M}) \cap \mathbf{1}^\perp$ be such that $u_n \to u$ and $v_n \to v$ in $L^2(\mathcal{M})$ as $n \to \infty$, and assume $\|u_n\|_{L^2} \leq 2\|u\|_{L^2}$ and $\|v_n\|_{L^2} \leq 2\|v\|_{L^2}$ for all n. Therefore

$$\begin{aligned}|\langle \varphi_{-t}^* u, v\rangle_{L^2}| &\leq |\langle \varphi_{-t}^*(u - u_n), v\rangle_{L^2}| + |\langle \varphi_{-t}^* u_n, v - v_n\rangle_{L^2}| + |\langle \varphi_{-t}^* u_n, v_n\rangle_{L^2}|\\ &\leq \|u - u_n\|_{L^2}\|v\|_{L^2} + 2\|u\|_{L^2}\|v - v_n\|_{L^2} + |\langle \varphi_{-t}^* u_n, v_n\rangle_{L^2}|.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary, and take n large enough so that $\|u - u_n\|_{L^2} < \varepsilon$ and $\|v - v_n\|_{L^2} < \varepsilon$, and t large enough such that $|\langle \varphi_{-t}^* u_n, v_n\rangle_{L^2}| < \varepsilon$. Thus for t large enough

$$|\langle \varphi_{-t}^* u, v\rangle_{L^2}| \leq (\|u\|_{L^2} + 2\|v\|_{L^2} + 1)\varepsilon,$$

and since ε was arbitrary this concludes the proof. □

Next we prove an auxiliary lemma saying that

Lemma 8.23. *The volume-preserving Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ is* transitive*, that is, there exists a point $x \in \mathcal{M}$ such that the orbit $\{\varphi_t x \mid t \in \mathbb{R}\} \subset \mathcal{M}$ is dense.*

Proof. Let $(U_j)_{j=1}^\infty \subset \mathcal{M}$ be a base of open sets for the topology on $\mathcal{M}$. Assume $x \in \mathcal{M}$ does not have a dense orbit: then there exists j such that $x \in S_j := \mathcal{M} \setminus \cup_{t\in\mathbb{R}}\varphi_t(U_j)$. Note that each S_j is flow-invariant, but not of full measure (since the measure of U_j is non-zero) so by ergodicity of $(\varphi_t)_{t\in\mathbb{R}}$ (see Theorem 8.21) we get that A_j has zero measure. Assuming the flow is not transitive, we get $\mathcal{M} = \cup_j A_j$ which is a contradiction since the measure of the right hand side is zero. This completes the proof. □

Finally, we are able to characterise Anosov flows that are not mixing by the following result, known as the *Anosov alternative*:

Theorem 8.24. *The volume-preserving Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ is* not *mixing with respect to Ω if and only if it is a suspension of an Anosov diffeomorphism $f : M \to M$ of a closed manifold M (recall suspensions were introduced in Example 8.2).*

Proof. If $\mathcal{M} = M \times [0, 1]/\sim$ is a suspension (up to diffeomorphism) of an Anosov diffeomorphism $f : M \to M$, then the function $F : \mathcal{M} \ni (x, \theta) \mapsto e^{2\pi i\theta} \in \mathbb{S}^1$ descends to a smooth function on the quotient $\mathcal{M}$ which satisfies $(X - 2\pi i)F = 0$, and hence the flow is not mixing by Lemma 8.22.

Conversely, assume the flow is not mixing, so by Lemma 8.22 there exists $u \in L^2(\mathcal{M})$ and $0 \neq \lambda \in \mathbb{R}$ such that $(X + i\lambda)u = 0$. Since $u \in L^2(\mathcal{M}) \subset L^1(\mathcal{M})$, by Lemma 8.20 we get $u \in C^\infty(\mathcal{M})$. Moreover, $X|u|^2 = Xu \cdot \overline{u} + u \cdot \overline{Xu} = (-i\lambda + i\lambda)|u|^2 = 0$, and so by ergodicity we get $|u|$ is constant, and up to rescaling we can assume that $|u| \equiv 1$. Integrating $\partial_t \varphi_t^* u = \varphi_t^* Xu = -i\lambda\varphi_t^* u$, we get $\varphi_t^* u = e^{-i\lambda t}u$ for all $t \in \mathbb{R}$, which implies that $u : \mathcal{M} \to \mathbb{S}^1$

is a submersion and so we may define codimension 1 smooth submanifolds $M_\theta := u^{-1}(e^{i\theta})$ for $\theta \in \mathbb{R}$. Observe that for all $t \in \mathbb{R}$

$$\varphi_t(M_\theta) = \varphi_t(u^{-1}(e^{i\theta})) = (\varphi_{-t}^* u)^{-1}(e^{i\theta}) = u^{-1}(e^{i(\theta-\lambda t)}) = M_{\theta-\lambda t}, \tag{8.31}$$

and so all $(M_\theta)_{\theta\in\mathbb{R}}$ are diffeomorphic to M_0.

Note that M_0 could have several connected components and let $N \subset M_0$ be one of them. By Lemma 8.23 there exists a point $x \in N$ with a dense orbit which in particular needs to cross N at some time. Hence there exists a minimal time $T > 0$ such that $\varphi_T x \in N$ and thus $\varphi_T(N) = N$. Note that $T = \frac{2\pi k}{\lambda}$ by (8.31), for some positive integer k. Define the map

$$\widetilde{D} : N \times \mathbb{R} \ni (x, \theta) \mapsto \varphi_\theta x \in \mathcal{M},$$

and note that it descends to a smooth map D on the mapping torus $S := N \times [0, T]/\sim$, where $(x, 1) \sim (\varphi_T x, 0)$, since $\widetilde{D}(x, \theta + T) = \varphi_T \widetilde{D}(x, \theta)$. Observe that D is injective: indeed, if $D(x, \theta) = D(y, \eta)$ for some $x, y \in N$ and $\theta, \eta \in [0, T)$, then $\varphi_{\theta-\eta} x = y \in N$ implying by the definition of T that $\theta - \eta \in T\mathbb{Z}$, and so $\theta = \eta$ implying further $x = y$. Note that D is also an immersion because N is transversal to the flow direction and hence D is an embedding which is necessarily onto because $\mathcal{M}$ is connected, hence a diffeomorphism onto $\mathcal{M}$.

If $\psi_t(x, \theta) = (x, \theta + t)$ is the vertical flow (descending to S), we observe that

$$\widetilde{D}\psi_t(x, \theta) = \widetilde{D}(x, \theta + t) = \varphi_{\theta+t} x = \varphi_t \widetilde{D}(x, \theta), \quad \theta, t \in \mathbb{R}, \quad x \in N$$

and hence D is a diffeomorphism that conjugates ψ_t and φ_t. It remains to show that $\varphi_T : N \to N$ is a volume-preserving Anosov diffeomorphism. To define a volume form on N, note that X is transversal to N and so the restriction of $\Omega_N := \iota_X \Omega$ to N is a volume form (here ι_X denotes contraction). Observe that φ_T preserves Ω_N since indeed $\varphi_T^* X = X$ and so $\varphi_T^* \Omega_N = \iota_X \varphi_T^* \Omega = \Omega_N$. To show the Anosov property, it suffices to show that $E_{u/s}(x) \subset T_x N$ for $x \in N$; then the Anosov property of φ_T follows from the one of $(\varphi_t)_{t\in\mathbb{R}}$. To prove the former, let $v \in E_s(x)$ for some $x \in N$ and write $v = aX(x) + u$, where $u \in T_x N$ and $a \in \mathbb{R}$. For any $m \in \mathbb{N}$, we see that

$$d\varphi_{mT}(x)(v) = a d\varphi_{mT}(x)(X(x)) + d\varphi_{mT}(x)(u) = aX(\varphi_{mT} x) + \underbrace{d\varphi_{mT}(x)(u)}_{\in T_{\varphi_{mT}(x)} N}.$$

If we let $m \to \infty$ the left hand side converges in norm to zero by the Anosov condition, and so by transversality of X and N we must have $a = 0$. Therefore $E_s(x) \subset T_x N$ and a similar argument shows that $E_u(x) \subset T_x N$, showing $T_x N = E_u(x) \oplus E_s(x)$ and completing the proof. $\square$

8.8. **Speed of mixing.**

9. Sinai-Ruelle-Bowen measure

The goal of this section is to introduce a natural measure, called *Sinai-Ruelle-Bowen (SRB) measure* which is a natural replacement for the volume form in the dissipative (non-volume preserving) case, and to give it a microlocal interpretation.

9.1. **Spectral approach to SRB measures.** In what follows, we will revisit the proof of Theorem 8.13 and some of its notation. Recall we took $T > 0$ and $\chi_T \in C^\infty([0,\infty);[0,1])$ with compact support, such that $\chi_T(t) = 1$ for $t \leq T$ and $\chi_T(t) = 0$ for $t \geq T+1$, and $-2 \leq \chi_T'(t) \leq 0$; recall (8.20) reads

$$(X+z)\underbrace{\int_0^\infty \chi_T(t)e^{-zt}\varphi_{-t}^*\,dt}_{Q(z):=} = \mathrm{Id} - \underbrace{\int_0^\infty (-\chi_T'(t))e^{-zt}\varphi_{-t}^*\,dt}_{R(z):=}. \tag{9.1}$$

Throughout the section will write $\psi := -\chi_T'$ and $\Pi(i\lambda)$ for the spectral projector at $\tau = 1$ of $R(i\lambda)$ (see the next lemma to see this is well defined). We will use the notation $\mathrm{Res}_X(z)$ to denote the space of resonant states of X at z. We will say that a bounded operator T on a Banach space $\mathcal{B}$ has *essential spectral radius* $r > 0$ if $T - \lambda$ is Fredholm for $|\lambda| > r$.

Lemma 9.1. *Let $\lambda \in \mathbb{R}$. The following holds:*

1. *$R(i\lambda)$ has essential spectral radius < 1.*
2. *$R(i\lambda)$ has no eigenvalues for $|z| > 1$.*
3. *$R(i\lambda)$ has an eigenvalue $\tau \in \mathbb{S}^1$ if and only if $i\lambda$ is a resonance of X, in which case $\tau = 1$ and the pole is of order ≤ 1.*
4. *As bounded operators on $\mathcal{H}$, we have*

$$\lim_{k\to\infty} R(i\lambda)^k = \Pi(i\lambda).$$

Proof. Item 1. By the proof of Theorem 8.13, if $T > 0$ is large then we may write $R(i\lambda) = D + K$ where K is smoothing and D satisfies $\|D\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{1}{2}$. The first claim then follows directly.

Item 2. For the second claim, observe firstly that for $|z| > \|R(i\lambda)\|_{\mathcal{H}\to\mathcal{H}}$ we have

$$(z - R(i\lambda))^{-1} = z^{-1}\sum_{k\geq 0} z^{-k}R(i\lambda)^k, \tag{9.2}$$

as an operator $\mathcal{H} \to \mathcal{H}$. Note that by the Analytic Fredholm Theorem and the first item, the resolvent $(z - R(i\lambda))^{-1}$ extends meromorphically to $|z| > r$ for some $r < 1$. Next, observe that by definition of $R(i\lambda)$ for any $f \in C^0(\mathcal{M})$ we have

$$\|R(i\lambda)f\|_{C^0} \leq \int_0^\infty (-\chi_T'(t))\|\varphi_{-t}^*f\|_{C^0}\,dt = \|f\|_{C^0}.$$

Therefore, similarly to (9.2), the Neumann series of the resolvent $(z - R(i\lambda)^{-1}$ converges and gives an operator $C^0(\mathcal{M}) \to C^0(\mathcal{M})$ for $|z| > 1$.

The two resolvents agree for $|z| \gg 1$ on $C^\infty(\mathcal{M})$ and $H^N(\mathcal{M})$ for some $N \gg 1$; similarly to the proof of Theorem 8.15 we derive that $R(i\lambda)$ has no eigenvalues for $|z| > 1$, proving the claim.

Item 3. Observe that

$$\begin{aligned}
R(i\lambda)^k &= \Big(\int_0^\infty (-\chi_T'(t))e^{-it\lambda}\varphi_{-t}^*\,dt\Big)^k \\
&= \int_{\mathbb{R}^k} (-\chi_T'(t_1))\cdots(-\chi_T'(t_k))e^{-i\lambda(t_1+\cdots+t_k)}\varphi^*_{-(t_1+\cdots+t_k)}\,dt_1\cdots dt_k \\
&= \int_0^\infty \psi^{(k)}(t)e^{-it\lambda}\varphi_{-t}^*\,dt,
\end{aligned} \tag{9.3}$$

where $\psi^{(k)}$ denotes the k-fold convolution. If $u, v \in C^\infty(\mathcal{M})$, it follows that

$$|\langle R(i\lambda)u, v\rangle_{L^2}| \leq \|u\|_{C^0}\|v\|_{C^0}R(0)^k\mathbf{1} = \|u\|_{C^0}\|v\|_{C^0},$$

where we used that $R(0)\mathbf{1} = \mathbf{1}$. Therefore if $|z| > 1$ and $u, v \in C^\infty(\mathcal{M})$

$$|\langle z(z - R(i\lambda)^{-1}u, v\rangle_{L^2}| \leq \sum_{k\geq 0} |z|^{-k}|\langle R(i\lambda)u, v\rangle_{L^2}| \leq (1-|z|^{-1})^{-1}\|u\|_{C^0}\|v\|_{C^0}.$$

Assume $(z-R(i\lambda))^{-1}$ has a pole of order ≥ 2 at $\tau \in \mathbb{S}^1$. By density, there exist $u, v \in C^\infty(\mathcal{M})$ such that $\langle (z-R(i\lambda))^{-1}u, v\rangle$ has a pole of degree ≥ 2 at τ; but this contradicts the preceding inequality and shows τ can only be a pole of degree ≤ 1.

For the other part of the third item, if u is a resonant state of $X + i\lambda$, then clearly $u = R(i\lambda)u$ is also an eigenfunction with eigenvalue 1 of $R(i\lambda)$. Conversely, observe $R(i\lambda)$ commutes with X and hence so does $\Pi(i\lambda)$; thus X acts on $\operatorname{ran}\Pi(i\lambda)$. Thus we may take $0 \neq u \in \mathcal{H}$ such that $R(i\lambda)u = \tau u$ and $(X + z)u = 0$ for some $z \in \mathbb{C}$. The latter equation implies $\varphi_{-t}^*u = e^{zt}u$ for $t \in \mathbb{R}$ and so

$$\tau u = R(i\lambda)u = \int_0^\infty \psi(t)e^{-i\lambda t}\varphi_{-t}^*u\,dt = u\cdot\widehat{\psi}(\lambda + iz),$$

which implies $\tau = \widehat{\psi}(\lambda + iz)$. Thus we have

$$1 = |\widehat{\psi}(\lambda + iz)| \leq \int_0^\infty e^{\mathrm{Re}(z)t}\psi(t)\,dt \leq \int_0^\infty \psi(t)\,dt = 1,$$

where we used that $\mathrm{Re}(z) \leq 0$ since there are no resonances with positive real part.

This implies that $\mathrm{Re}(z) = 0$ and moreover that $z = i\lambda$ and $\tau = 1$, completing the proof of the third item.

Item 4. For the fourth statement, by the first and third points we write $R(i\lambda) = \Pi(i\lambda) + K(i\lambda)$, where $K(i\lambda)$ has spectral radius < 1, i.e. $z - K(i\lambda)$ is invertible for $|z| > r$ for some $r < 1$. By definition of a spectral projector, $\Pi(i\lambda)$ commutes with $R(i\lambda)$ and so it commutes with $K(i\lambda)$ as well. Moreover

$$\Pi(i\lambda)K(i\lambda) = K(i\lambda)\Pi(i\lambda) = (R(i\lambda) - \Pi(i\lambda))\Pi(i\lambda) = 0,$$

where in the last equality we used $\Pi(i\lambda)^2 = \Pi(i\lambda)$ and $R(i\lambda)$ fixes elements of $\operatorname{ran}\Pi(i\lambda)$. By Gelfand's spectral radius formula, there exist $0 < r < 1$ and N_0 such that

$$\|K(i\lambda)^N\|_{\mathcal{H}\to\mathcal{H}} \leq r^N, \quad N \geq N_0.$$

It follows that $R(i\lambda)^k = \Pi(i\lambda) + K(i\lambda)^k \to \Pi(i\lambda)$ as $k \to \infty$, completing the proof. $\square$

Note that the pairing

$$\mathcal{H}^{m(x,\xi)} \times \mathcal{H}^{-m(x,\xi)} \ni (u,v) \mapsto \langle u, v\rangle_{L^2}$$

is well-defined and non-degenerate (by the properties of A_m). Via this pairing we may identify the dual $(\mathcal{H}^{m(x,\xi)})'$ with $\mathcal{H}^{-m(x,\xi)}$. Recall we fixed a probability volume form Ω and wrote $\operatorname{div}(X)$ for the divergence of X with respect to Ω, i.e. $\mathcal{L}_X \Omega = \operatorname{div}(X)\Omega$; we will write $X^* = -X - \operatorname{div}(X)$ for the L^2-adjoint of X. Then, we have

Lemma 9.2. *The projector $\Pi(i\lambda)$ does not depend on the choices made in defining $R(i\lambda)$. More precisely, if $v_1, \dots, v_J \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ and $w_1, \dots, w_J \in \mathcal{D}'_{E_s^*}(\mathcal{M})$ form a basis of resonant states of X at $i\lambda$ and of X^* at $-i\lambda$ such that $\langle v_i, w_j\rangle_{L^2} = \delta_{ij}$ for $i,j = 1, \dots, J$, then*

$$\Pi(i\lambda)(\bullet) = \sum_{i=1}^J v_i \langle \bullet, w_i\rangle.$$

Proof. By Theorem 8.15 and Lemma 9.1, we may write $\Pi(i\lambda)(\bullet) = \sum_{i=1}^J v_i\langle \bullet, \widetilde{w}_i\rangle_{L^2}$ for some $\widetilde{w}_i \in \mathcal{H}^{-m(x,\xi)}$. Again by Lemma 9.1 (item 3) we have $(X + i\lambda)\Pi(i\lambda) \equiv \Pi(i\lambda)(X + i\lambda) \equiv 0$. This implies that $(X^* - i\lambda)\widetilde{w}_i = 0$ for all $i = 1, \dots, J$.

Now, write $R_X(z) := R(z)$ to emphasise the dependence on X, and write

$$R_{X^*}(z) = \int_0^\infty \psi(t) e^{\int_0^t \varphi_s^* \operatorname{div}(X)\, ds} e^{-zt} \varphi_t^*\, dt. \tag{9.4}$$

One can similarly define $Q_{X^*}(z)$ so that $Q_{X^*}(z)(X^* + z) = \operatorname{Id} - R_{X^*}(z)$ and argue as in the proof of Theorem 8.13 to show that the resolvent $(X^* + z)^{-1}$ admits a suitable meromorphic continuation on $\mathcal{H}^{-m(x,\xi)}$. In fact, as follows from definition (9.4) we have $(R_X(i\lambda))^* = R_{X^*}(-i\lambda)$ in the sense of L^2 adjoints:

$$\langle R_X(i\lambda)u, v\rangle_{L^2} = \langle u, R_{X^*}(-i\lambda)v\rangle_{L^2}, \quad u \in \mathcal{H}^{m(x,\xi)}, \quad v \in \mathcal{H}^{-m(x,\xi)}.$$

Therefore, also

$$\Pi(i\lambda)^* = \Pi_{X^*}(-i\lambda), \tag{9.5}$$

where $\Pi_{X^*}(-i\lambda)$ denotes the spectral projector onto $\tau = 1$ of $R_{X^*}(-i\lambda)$. It follows that for any smooth u, v

$$\langle \Pi(i\lambda)u, v\rangle_{L^2} = \sum_{i=1}^J \langle v_i, v\rangle\langle u, \widetilde{w}_i\rangle = \langle u, \sum_{i=1}^J \widetilde{w}_i\langle v, v_i\rangle\rangle,$$

that is, $\Pi_{X^*}(-i\lambda) = \sum_{i=1}^J \widetilde{w}_i\langle v, v_i\rangle$. This implies that $(\widetilde{w}_i)_{i=1}^J$ is a basis of resonant states of X^* at $-i\lambda$, and in fact $w_i = \widetilde{w}_i$ for all i. □

The following theorem constructs the SRB measures and gives their microlocal interpretation. We will write $\langle \bullet, \bullet\rangle$ for the distributional pairing (to be distinguished from the L^2 pairing $\langle \bullet, \bullet\rangle_{L^2}$ which is sesquilinear).

Theorem 9.3. *The following points hold:*

1. *For any $v \in C^\infty(\mathcal{M}, \mathbb{R}_{>0})$,*

$$\mu_v : C^\infty(\mathcal{M}) \ni u \mapsto \langle \Pi(0)u, v\rangle$$

is a non-negative Radon measure invariant by the flow, called SRB measure*.*

2. *We have*
$$\mathrm{span}\{\mu_v \mid v \in C^\infty(\mathcal{M}, \mathbb{R}_{>0})\} = \Pi(0)^*(C^\infty(\mathcal{M})),$$
where $\Pi(0)^*$ *is the* L^2 *adjoint of* $\Pi(0)$. *Equivalently,* $\Pi(0)^*(C^\infty(\mathcal{M}))$ *can be interpreted as the set of complex measures with wavefront set in* E_s^* *invariant by the flow.*
3. *For any* $\lambda \in \mathbb{R}$ *and* $v \in C^\infty(\mathcal{M})$, *the map*
$$\mu_v^\lambda : C^\infty(\mathcal{M}) \ni u \mapsto \langle \Pi(i\lambda)u, v\rangle$$
is a finite complex valued measure. These measures satisfy the invariance $\mu_v^\lambda(Xu) = -i\lambda\mu_v^\lambda(u)$ *for any* $u \in C^\infty(\mathcal{M})$, *the set* $\{\mu_v^\lambda \mid v \in C^\infty(\mathcal{M})\}$ *coincides with the set of finite complex measures with wavefront set in* E_s^* *which are invariant in the above sense, and* μ_v^λ *are absolutely continuous with respect to* $\mu := \mu_{\mathbf{1}}$.
4. *The measures* μ_v *are Cesaro averages of smooth measures, that is for any* $u \in C^\infty(\mathcal{M})$ *we have*
$$\mu_v(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \varphi_{-t}^* u, v\rangle \, dt.$$

Proof. Item 1. Observe that for $u \in C^0(\mathcal{M})$
$$\|R(0)^k u\|_{C^0} \leq R(0)^k \mathbf{1} \cdot \|u\|_{C^0} = \|u\|_{C^0},$$
where we used the formula (9.3), and if additionally $u \geq 0$, then $R(0)^k u \geq 0$. It follows that
$$|\mu_v(u)| = |\langle \Pi(0)u, v\rangle_{L^2}| \leq \limsup_{k\to\infty} |\langle R(0)^k u, v\rangle_{L^2}| \leq \|u\|_{C^0}\|v\|_{C^0}, \quad \forall u \in C^0(\mathcal{M}),$$
and so μ_v defines a Radon measure. It is invariant as $\mu_v(\varphi_t^* u) = \mu_v(u)$ for any real t, since $\Pi(0)$ commutes with φ_t^* and the range of $\Pi(0)$ consists of flow-invariant distributions.

Item 2. Given an SRB measure μ_v, we simply observe that
$$\mu_v(u) = \langle \Pi(0)u, v\rangle_{L^2} = \langle u, \Pi(0)^* v\rangle_{L^2}.$$
It is left to observe that
$$\overline{\Pi(0)^* v} = \overline{\Pi_{X^*}(0)v} = \Pi_{X^*}(0)v = \Pi(0)^* v,$$
where we used the characterisation of $\Pi(0)^*$ obtained in the proof of Lemma 9.2 and the fact that v is real-valued. For the other inclusion, it simply suffices to observe that any $v \in C^\infty(\mathcal{M})$ splits as $v = v_1 + iv_2$ where $v_i \in C^\infty(\mathcal{M}, \mathbb{R})$ and furthermore that $v_i = v_i^+ - v_i^-$ for some non-negative smooth functions $v_i^\pm$. The wavefront set characterisation follows again from $\Pi(0)^* = \Pi_{X^*}(0)$ (and the wavefront set characterisation of resonant states, see Lemma 8.16).

Item 3. That μ_v^λ are finite complex measures follows from Lemma 9.1, Item 4, and that $\|R(i\lambda)u\|_{C^0} \leq \|u\|_{C^0}$, see (9.3) (this is similar to Item 1 above). Their invariance is a consequence of the fact that $0 \equiv (X + i\lambda)\Pi(i\lambda) = \Pi(i\lambda)(X + i\lambda)$, and the wavefront set interpretation follows from
$$\mathrm{span}\{\mu_v \mid v \in C^\infty(\mathcal{M})\} = \Pi(i\lambda)^*(C^\infty(\mathcal{M})),$$
obtained similarly to Item 2, and the fact that $\Pi(i\lambda)^* = \Pi_{X^*}(-i\lambda)$ (which follows from (9.5)). Absolute continuity with respect to $\mu_{\mathbf{1}}$ can be seen as
$$|\langle R(i\lambda)^k u, v\rangle| \leq \langle R(0)^k |u|, \mathbf{1}\rangle \cdot \|v\|_{C^0}, \quad \forall u \in C^\infty(\mathcal{M}), \quad \forall k \in \mathbb{N},$$

and so by taking the limit $k \to \infty$ we get $|\mu_v^\lambda(u)| \leq \|v\|_{C^0}\mu_{\mathbf{1}}(|u|)$. Approximating indicator functions by smooth functions and using the Dominated Convergence Theorem we get $|\mu_v^\lambda(A)| \leq \|v\|_{C^0}\mu_{\mathbf{1}}(A)$ for any closed set A, and hence for any Borel set A, which proves the claim.

Item 4. We divide the proof into several steps.

Step 1: average over smooth functions. Let $\omega \in C_0^\infty(\mathbb{R}_{>0})$ be a cut-off function such that $\int_0^\infty \omega(x)\,dx \neq 0$. We claim that, for any $u \in C^\infty(\mathcal{M})$ and $v \in C^\infty(\mathcal{M}, \mathbb{R}_{>0})$

$$\frac{1}{T\int_0^\infty \omega(x)\,dx}\int_0^\infty \omega(\tfrac{t}{T})\langle \varphi_{-t}^* u, v\rangle\,dt \to_{T\to\infty} \mu_v(u). \tag{9.6}$$

We will without loss of generality assume ω integrates to 1. Let $\psi \in C_0^\infty((-\frac{1}{2},\frac{1}{2}))$ be non-negative, even, and such that $\int_{\mathbb{R}}\psi(x)\,dx = 1$. Let $\sigma > 1/2$ and choose χ_T in the construction of Theorem 8.13 such that $-\chi_T'(t) = -\chi' = \psi(t-\sigma)$; from now on σ is fixed to be large so that Theorem 8.13 applies. Denote by $R_\sigma(z)$ the remainder term in (9.1). We claim that

$$\mu_v(u) = \lim_{T\to\infty}\frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\langle R_\sigma(0)^k u, v\rangle \tag{9.7}$$

Indeed, we have

$$\begin{aligned}-\mu_v(u) + \frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\langle R_\sigma(0)^k u, v\rangle &= \frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})(\langle R_\sigma(0)^k u, v\rangle - \mu_v(u)) - \mu_v(u)\Big(1 - \frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\Big),\\ &= o(1), \quad T\to\infty,\end{aligned}$$

where the factor in the first bracket goes to zero as $T\to\infty$ by Lemma 9.1, Item 4, while the second one goes to zero using Riemann sums.

Step 2: a reduction. Next, we claim that

$$\frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\langle R_\sigma(0)^k u, v\rangle - \frac{1}{\sigma T}\int_0^\infty \omega(\tfrac{t}{\sigma T})\langle\varphi_{-t}^* u, v\rangle\,dt = o(1), \quad T\to\infty,$$

which would immediately prove the initial claim (9.6) using (9.7). For this, using the iterated resolvent formula (9.3) it suffices to show

$$\frac{1}{T}\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\psi_\sigma^{(k)}(t) - \frac{1}{\sigma T}\omega(\tfrac{t}{\sigma T}) = o_{L^1(\mathbb{R})}(1), \quad T\to\infty.$$

By a change of variables $t\mapsto tT$, this is equivalent to

$$\underbrace{\sum_{k\in\mathbb{N}}\omega(\tfrac{k}{T})\psi_\sigma^{(k)}(tT) - \tfrac{1}{\sigma}\omega(\tfrac{t}{\sigma})}_{f_T(t):=} = o_{L^1(\mathbb{R})}(1), \quad T\to\infty.$$

Now, since the support of convolutions is included in the sum of the supports, we have for

$$\operatorname{supp}\psi_\sigma^{(k)}(\bullet T) \subset \frac{k}{T}\operatorname{supp}\psi_\sigma \subset \frac{k}{T}[\sigma - \tfrac{1}{2}, \sigma + \tfrac{1}{2}],$$

and so $\operatorname{supp} f_T$ is included in a bounded set independently of T (since $\frac{k}{T}\in\operatorname{supp}\omega$). Therefore, to prove that $f_T \to \frac{1}{\sigma}\omega(\frac{\bullet}{\sigma})$ in $L^1(\mathbb{R})$ as $T\to\infty$, it suffices to show convergence in $L^2(\mathbb{R})$; in turn, it suffices to show convergence of Fourier transforms.

Observe by the assumptions that $\widehat{\psi}$ is real-valued and that we may write its Taylor expansion as

$$\widehat{\psi}(\xi) = 1 - \frac{\xi^2}{2}\int_{\mathbb{R}} t^2\psi(t)\,dt + \mathcal{O}(|\xi|^4), \quad \xi \to 0. \tag{9.8}$$

where the terms odd in ξ vanish thanks to evenness of ψ and the first term is 1 thanks to the fact that ψ integrates to 1. We conclude that

$$\widehat{\psi}(\xi) = 1 + \mathcal{O}(|\xi|^2), \quad \xi \to 0. \tag{9.9}$$

Finally, we record one more property of $\widehat{\psi}$; on one hand, since ψ is compactly supported, $\widehat{\psi} = \mathcal{O}(|\xi|^{-\infty})$ as $|\xi| \to \infty$. One the other hand, $|\widehat{\psi}(\xi)| \leq 1$ with equality at $\xi = 0$ and so combining this with the Taylor expansion (9.8), we get for some $c > 0$ that

$$|\widehat{\psi}(\xi)| \leq (1 + c|\xi|^2)^{-1}, \quad \xi \in \mathbb{R}. \tag{9.10}$$

We compute for fixed $\xi \in \mathbb{R}$:

$$\begin{aligned}\widehat{f}_T(\xi) &= \frac{1}{T}\sum_{k\in\mathbb{N}} \omega(\tfrac{k}{T})\widehat{\psi}(\tfrac{\xi}{T})^k e^{-ik\sigma\frac{\xi}{T}} = \frac{1}{T}\sum_{k\in\mathbb{N}} \omega(\tfrac{k}{T})(1 + \mathcal{O}(|\tfrac{\xi}{T}|^2))^k e^{-ik\sigma\frac{\xi}{T}}\\ &= \frac{1}{T}\sum_{k\in\mathbb{N}} \omega(\tfrac{k}{T})(1 + k\mathcal{O}(|\tfrac{\xi}{T}|^2))e^{-i\xi\sigma\frac{k}{T}}\\ &= \int_0^\infty \omega(t)e^{-i\sigma\xi t}\,dt + o(1) = \tfrac{1}{\sigma}\widehat{\omega(\tfrac{\bullet}{\sigma})}(\xi) + o(1),\end{aligned} \tag{9.11}$$

as $T \to \infty$, where in the second equality we used (9.9), in the third one we used that $|\frac{\xi}{T}|$ is small, and in the last line we used Riemann sums and that $\frac{k}{T} = \mathcal{O}(1)$ as ω has compact support. This shows pointwise convergence $\widehat{f}_T \to \frac{1}{\sigma}\widehat{\omega(\frac{\bullet}{\sigma})}$, and in fact, uniform convergence on compact sets.

Step 3: large frequencies. Let $\varepsilon > 0$ such that $\operatorname{supp}\omega \subset (\varepsilon, \infty)$, and fix $\delta \in (0, \frac{1}{2})$. Then:

$$\begin{aligned}\int_{|\xi|\geq T^{\frac{1}{2}+\delta}} |\widehat{f}_T(\xi)|^2\,d\xi &\leq \frac{\mathcal{O}(T)}{T^2}\sum_{k\in\mathbb{N}} |\omega(\tfrac{k}{T})|^2 \int_{|\xi|\geq T^{\frac{1}{2}+\delta}} |\widehat{\psi}(\tfrac{\xi}{T})|^{2k}\,d\xi\\ &\leq \mathcal{O}(1)\sum_{k\in\mathbb{N}} |\omega(\tfrac{k}{T})|^2 \int_{|\xi|\geq T^{-\frac{1}{2}+\delta}} |\widehat{\psi}(\xi)|^{2k}\,d\xi\\ &\leq \mathcal{O}(T)\|\omega\|_{C^0}\int_{\xi\geq T^{-\frac{1}{2}+\delta}} (1 + c|\xi|^2)^{-2\varepsilon T}\,d\xi\\ &\leq \mathcal{O}(T)\|\omega\|_{C^0}\int_{x\geq cT^{-1+2\delta}} (1 + x)^{-2\varepsilon T}x^{-\frac{1}{2}}\,dx\\ &\leq \mathcal{O}(T^{\frac{3}{2}-\delta})\|\omega\|_{C^0}\int_{x\geq cT^{-1+2\delta}} (1 + x)^{-2\varepsilon T}\,dx\\ &\leq \mathcal{O}(T^{\frac{1}{2}-\delta})\|\omega\|_{C^0}(1 + cT^{-1+2\delta})^{-2\varepsilon T}\end{aligned}$$

Set $y^{-1} := cT^{1-2\delta}$; then

$$(1 + cT^{-1+2\delta})^{-2\varepsilon T} = (1 + y^{-1})^{-yR} \leq 2^{-R}, \quad R = 2\varepsilon c^{\frac{1}{1-2\delta}} y^{\frac{2\delta}{1-2\delta}},$$

for T large enough. We conclude that

$$\int_{|\xi|\geq T^{\frac{1}{2}+\delta}} |\widehat{f}_T(\xi)|^2\, d\xi = o(1), \quad T\to\infty. \tag{9.12}$$

Step 4: intermediate frequencies. We now deal with frequencies $1\leq|\xi|\leq T^{\frac{1}{2}+\delta}$; for this we perform a discrete integration by parts to gain some decay in ξ variable for $\widehat{f}_T$. Let $\rho\in C_0^\infty(\mathbb{R}_{>0})$, and set

$$a_k^0 := \rho(\tfrac{k}{T}), \quad b_k^0 := e^{-i\sigma\xi\frac{k}{T}}, \quad k\in\mathbb{Z}.$$

Then inductively we set for $m\geq 1$ and $k\in\mathbb{Z}$

$$b_k^m := b_k^{m-1}\frac{e^{-i\frac{\sigma\xi}{T}}}{1-e^{-i\frac{\sigma\xi}{T}}} = e^{-i\frac{\sigma\xi}{T}}\Big(\frac{e^{-i\frac{\sigma\xi}{T}}}{1-e^{-i\frac{\sigma\xi}{T}}}\Big)^m,$$

$$a_k^m := a_k^{m-1} - a_{k+1}^{m-1} = \sum_{i=0}^m \rho(\tfrac{k+i}{T})(-1)^i\binom{m}{i}.$$

Let $\operatorname{supp}\rho=[\rho_1,\rho_2]$; we observe the following immediate properties of a_k^m and b_k^m:

$$a_k^m = 0, \quad \text{if} \quad \frac{k+m}{T}<\rho_1 \quad \text{or} \quad \frac{k}{T}>\rho_2, \quad b_k^m = b_{k-1}^{m+1}-b_k^{m+1}, \quad k\in\mathbb{Z}, \quad m\in\mathbb{N}.$$

Using the latter identity and the definition of a_k^m we get

$$\sum_{k\in\mathbb{Z}} a_k^m b_k^m = \sum_{k\in\mathbb{Z}} a_k^m(b_{k-1}^{m+1}-b_k^{m+1}) = -\sum_{k\in\mathbb{Z}}(a_k^m - a_{k+1}^m)b_k^{m+1} = -\sum_{k\in\mathbb{Z}} a_k^{m+1}b_k^{m+1}. \tag{9.13}$$

Next, observe that since ρ is smooth, by an iterated application of the Mean Value Theorem we get

$$|a_k^m| \leq C\|\rho\|_{C^m}T^{-m}, \quad k\in\mathbb{Z}, \quad m\geq 0,$$

where $C=C(m)>0$ is constant depending only on m. Using the estimate $|1-e^{ix}|\geq C|x|$ valid for $|x|\leq\frac{\pi}{2}$ and some constant $C>0$, we get

$$|b_k^m| \leq C\frac{T^m}{|\sigma\xi|^m}, \quad k\in\mathbb{Z}, \quad m\geq 0.$$

Combining the two preceding estimates and applying (9.13) we get:

$$\Big|\sum_{k\in\mathbb{N}}\rho(\tfrac{k}{T})e^{-i\sigma\xi\frac{k}{T}}\Big| \leq (\rho_2-\rho_1+1)T\min(\|\rho\|_{C^0}, C\|\rho\|_{C^m}(|\sigma\xi|)^{-m}), \quad m\geq 1. \tag{9.14}$$

Let us apply (9.14) with $\rho(x)=\omega(x)e^{xT\log(\widehat{\psi}(\frac{\xi}{T}))}$, where we notice the logarithm is well-defined as $\widehat{\psi}$ is real-valued and $|\widehat{\psi}(\frac{\xi}{T})-1| = \mathcal{O}(|\frac{\xi}{T}|^2)=\mathcal{O}(T^{2\delta-1})$ is small if T is taken large enough (we used (9.9)). Similarly for T large enough there is a constant $C>0$ such that

$$|T\log(\widehat{\psi}(\tfrac{\xi}{T}))| \leq C\frac{|\xi|^2}{T}.$$

By the preceding estimate we get for some $C>0$, and using that $\widehat{\psi}(\frac{\xi}{T})<0$ for T large enough

$$\|\rho\|_{C^m} \leq C\|\omega\|_{C^m}\left(1+\frac{|\xi|^{2m}}{T^m}\right)e^{xT\log(\widehat{\psi}(\frac{\xi}{T}))} \leq C\|\omega\|_{C^m}\left(1+\frac{|\xi|^{2m}}{T^m}\right). \tag{9.15}$$

Finally, we deduce from (9.15) and (9.14) that, for $|\xi| \leq T^{\frac{1}{2}+\delta}$

$$|\widehat{f}_T(\xi)| \leq C\|\rho\|_{C^m}|\xi|^{-m} \leq C\left(1+\frac{|\xi|^{2m}}{T^m}\right)|\xi|^{-m}\sigma^{-m} \leq C\sigma^{-m}\left(|\xi|^{-m}+T^{-m(\frac{1}{2}-\delta)}\right). \quad (9.16)$$

Then we have

$$\|\widehat{f}_T - \tfrac{1}{\sigma}\widehat{\omega}(\tfrac{\bullet}{\sigma})\|_{L^2}^2 = \int_{|\xi|\leq 1} |\widehat{f}_T(\xi) - \tfrac{1}{\sigma}\widehat{\omega}(\tfrac{\xi}{\sigma})|^2\, d\xi + \int_{1\leq|\xi|\leq T^{\frac{1}{2}+\delta}} |\widehat{f}_T(\xi) - \tfrac{1}{\sigma}\widehat{\omega}(\tfrac{\xi}{\sigma})|^2\, d\xi$$
$$+ \int_{|\xi|\geq T^{\frac{1}{2}+\delta}} |\widehat{f}_T(\xi) - \tfrac{1}{\sigma}\widehat{\omega}(\tfrac{\xi}{\sigma})|^2\, d\xi = o(1),$$

as $T \to \infty$, where the first summand equals $o(1)$ by (9.11), the second equals $o(1)$ by (9.16) and the Dominated Convergence Theorem, and the last summand is equal to $o(1)$ since $\frac{1}{\sigma}\omega(\frac{\bullet}{\sigma})$ has a rapidly decaying Fourier transform and using (9.12). This completes the proof of (9.6).

Step 5: conclusion. Let $f \in L^1(\mathbb{R}_{\geq 0})$ such that $\int_0^\infty f(x)\, dx \neq 0$. We claim that

$$\mathcal{I}_T(f) := \frac{1}{T\int_0^\infty f(x)\, dx}\int_0^\infty f(\tfrac{t}{T})\langle\varphi_{-t}^* u, v\rangle\, dt \to_{T\to\infty} \mu_v(u). \quad (9.17)$$

Without loss of generality we may assume that $\int_0^\infty f(x)\, dx = 1$. By density of smooth functions, choose $(\omega_i)_{i=1}^\infty \subset C_0^\infty(\mathbb{R}_{>0})$ such that $\int_0^\infty \omega_i(x)\, dx = 1$ and $\omega_i \to f$ in L^1 as $i \to \infty$. Observe that

$$|\mathcal{I}_T(f) - \mathcal{I}_T(\omega_i)| \leq \int_0^\infty |f(t) - \omega_i(t)| \cdot \langle\varphi_{-tT}^* u, v\rangle\, dt \leq \|u\|_{C^0}\|v\|_{C^0}\|f-\omega_i\|_{L^1}. \quad (9.18)$$

Let $\varepsilon > 0$. Pick i such that $\|f-\omega_i\|_{L^1} < \varepsilon$ and T_0 such that $T \geq T_0$ implies $|\mathcal{I}_T(\omega_i) - \mu_v(u)| < \varepsilon$. Then we have for $T \geq T_0$

$$|\mu_v(u) - \mathcal{I}_T(f)| \leq |\mu_v(u) - \mathcal{I}_T(\omega_i)| + |\mathcal{I}_T(\omega_i) - \mathcal{I}_T(f)| \leq \varepsilon(1 + \|u\|_{C^0}\|v\|_{C^0}),$$

where in the second inequality we used (9.18). Since $\varepsilon > 0$ was arbitrary, this proves (9.17); the main clam then follows by taking f to be the indicator function of the interval $[0,1]$. $\square$

Remark 9.4. There might be an easier way to deal with Item 4 which does not go through the complicated estimations of Fourier transforms. To do that, it would help to show that $\{\varphi_{-t}^* \mid t \geq 0\}$ are *uniformly* bounded as maps $\mathcal{H}^+ \to \mathcal{H}^+$ (as in Butterley-Liverani, but they use different spaces).

Before characterising the spectrum on the imaginary axis, we deduce a regularity result analogous to Lemma 8.20. We will denote by μ the SRB measure introduce in Theorem 9.3, Item 3.

Lemma 9.5. *Let $\lambda \in \mathbb{R}$ and $u \in L^1(\mathcal{M}, \mu)$ such that $\varphi_t^* u(x) = e^{-i\lambda t}u(x)$ for all $t \in \mathbb{R}$ and μ-a.e. $x \in \mathcal{M}$. Then there exists a resonant $\widetilde{u}$ of X at $i\lambda$ such that $\widetilde{u}\mu = u\mu$.*

Proof. The proof is similar to the one of Lemma 8.20. We will assume that $\lambda = 0$; the case of $\lambda \neq 0$ follows similarly. Let $u_n \in C^\infty(\mathcal{M})$ be such that $u_n \to u$ in $L^1(\mathcal{M}, \mu)$ as $n \to \infty$ and set $v_n := \Pi(0)u_n \in \mathcal{H}^+$. We divide the proof in two steps.

Step 1. Assume $\|v_n\|_{\mathcal{H}^+} \to \infty$. Since $v_n \in \ker X|_{\mathcal{H}^+}$ which is a finite dimensional space, we may assume $\frac{v_n}{\|v_n\|_{\mathcal{H}^+}} \to v_\infty$ in $\mathcal{H}^+$ with $\|v_\infty\|_{\mathcal{H}^+} = 1$. Then for an arbitrary $\psi \in C^\infty(\mathcal{M})$

$$\langle v_\infty, \psi\rangle = \langle v_\infty - \tfrac{v_n}{\|v_n\|_{\mathcal{H}^+}}, \psi\rangle + \tfrac{1}{\|v_n\|_{\mathcal{H}^+}}\langle u_n, \Pi_{X^*}(0)\psi\rangle.$$

Let $\varepsilon > 0$; since $v_\infty \to \frac{v_n}{\|v_n\|_{\mathcal{H}^+}}$ in $\mathcal{D}'(\mathcal{M})$ as $n \to \infty$, we thus have for all n large enough that $|\langle v_\infty - \frac{v_n}{\|v_n\|_{\mathcal{H}^+}}, \psi\rangle| < \varepsilon$. For the second term, by Lemma 9.2, we have

$$\Pi_{X^*}(0)(\bullet) = \sum_{i=1}^J w_i \langle \widetilde{v}_i, \bullet\rangle,$$

where similarly to Lemma 9.2 we denote by $(w_i)_{i=1}^J$ a basis of the resonant states of X^* at 0 and by $(\widetilde{v}_i)_{i=1}^J$ a basis of resonant states of X at 0, such that $\langle \widetilde{v}_i, w_j\rangle = \delta_{ij}$ for $i, j = 1, \dots, J$. Thus

$$|\langle u_n, \Pi_{X^*}(0)\psi\rangle| \leq \sum_{i=1}^J |\langle u_n, w_i\rangle||\langle \psi, \widetilde{v}_i\rangle| \leq \mathcal{O}(1)\sum_{i=1}^J |\langle u_n, w_i\rangle| = \mathcal{O}(1), \quad n \to \infty,$$

where in the first inequality we used that for each i we have $|\langle \psi, \widetilde{v}_i\rangle| = \mathcal{O}(1)$, and in the second inequality we used that for all n large enough $\|u_n\|_{L^1(\mathcal{M},\mu)} \leq 2\|u\|_{L^1(\mathcal{M},\mu)}$, as well as that the (possibly) complex measure defined by w_i (via Theorem 9.3) is absolutely continuous with respect to μ with bounded density. For all n large enough, we have $\|v_n\|_{\mathcal{H}^+}^{-1} < \varepsilon$ and so $|\langle v_\infty, \psi\rangle| = \mathcal{O}(\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, this shows $v_\infty = 0$ contradicting $\|v_\infty\|_{\mathcal{H}^+} = 1$.

Step 2. By Step 1 we may assume $(\|v_n\|_{\mathcal{H}^+})_{n=1}^\infty$ is bounded; again, since $\ker X|_{\mathcal{H}^+}$ is finite dimensional we may assume $v_n \to v_\infty$ in $\mathcal{H}^+$. We claim that $\widetilde{u} := v_\infty$ satisfies the conditions of the lemma. Notice that $z\mathbf{R}_+(z)u = u$ for $\operatorname{Re}(z) > 0$, where by slight abuse of notation $\mathbf{R}_+(z) : L^1(\mathcal{M},\mu) \to L^1(\mathcal{M},\mu)$ is defined by the resolvent formula as in (8.24) and satisfies the bounds as in (8.25). We compute, denoting $\langle \bullet, \bullet\rangle_\mu$ the complex bilinear form defined by integrating against μ:

$$\langle u - v_\infty, \psi\rangle_\mu = \langle z\mathbf{R}_+(z)(u - u_n), \psi\rangle_\mu + \langle z\mathbf{R}_+(z)u_n - v_n, \psi\rangle_\mu + \langle v_n - v_\infty, \psi\rangle_\mu,$$

where the terms including v_n are well-defined since $\mu = f\Omega$ for some $f \in \mathcal{H}^-$ and the integration pairing is bounded on $\mathcal{H}^+ \times \mathcal{H}^-$. We analyse the three terms on the right hand side; let $\varepsilon > 0$. Since $v_n \to v_\infty$ in $\mathcal{D}'(\mathcal{M})$ as $n \to \infty$ for all n large enough $|\langle v_n - v_\infty, \psi\rangle_\mu| < \varepsilon$.

For the first term we write, for any $z \in \mathbb{R}_{>0}$

$$|\langle z\mathbf{R}_+(z)(u - u_n), \psi\rangle_\mu| \leq \|u - u_n\|_{L^1(\mathcal{M},\mu)}\|\psi\|_{C^0} < \varepsilon,$$

for all n large enough so that $\|u - u_n\|_{L^\infty(\mathcal{M},\mu)} < \varepsilon\|\psi\|_{C^0}^{-1}$; we used (8.25) in the first inequality.

For the second term, we fix n large enough so that the two estimates previously obtained hold. Observe that $z\mathbf{R}_+(z)u_n - v_n = z\mathbf{R}_+^{\mathrm{hol}}(z)u_n$ (using the expansion at zero (8.27)). Thus

$$|\langle z\mathbf{R}_+(z)u_n - v_n, \psi\rangle_\mu| = |z||\langle \mathbf{R}_+^{\mathrm{hol}}(z)u_n, f\psi\rangle| \leq |z|\|\mathbf{R}_+^{\mathrm{hol}}(z)\|_{\mathcal{H}^+\to\mathcal{H}^+}\|u_n\|_{\mathcal{H}^+}\|f\psi\|_{\mathcal{H}^-} < \varepsilon,$$

if $|z|$ is small enough depending on n and ε, where we wrote $\mu = f\Omega$ for some $f \in \mathcal{H}^-$, and using that $\mathbf{R}_+^{\mathrm{hol}}(z)$ is locally bounded in z, and that the integration pairing on $\mathcal{H}^+ \times \mathcal{H}^-$ is bounded.

Collecting the above facts we get $|\langle u - v_\infty, \psi\rangle_\mu| < 3\varepsilon$; since $\varepsilon > 0$ was arbitrary, the conclusion of the lemma follows. $\square$

For $\lambda \in \mathbb{R}$ denote by $\mathcal{M}^\lambda$ the vector space of all complex measures μ_v^λ defined in Theorem 9.3, Item 3. We will deduce a correspondence between $\mathcal{M}^\lambda$ and resonant states of X on the imaginary axis, as well as μ-measurable invariant bounded functions.

Lemma 9.6. *Let $\lambda \in \mathbb{R}$ and $\mu^\lambda \in \mathcal{M}^\lambda$; write $\mu^\lambda = u\mu$ for some $u \in L^\infty(\mathcal{M}, \mu)$. Then*

1. *μ^λ (and u) may be identified with a resonant state $\widetilde{u} \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ satisfying $(X + i\lambda)\widetilde{u} = 0$ in the sense that $u\mu = \widetilde{u}\mu$, i.e.*
$$\int_{\mathcal{M}} \psi \, d\mu_v^\lambda = \int_{\mathcal{M}} \psi \widetilde{u} \, d\mu, \quad \forall \psi \in C^\infty(\mathcal{M}).$$
Moreover, the map $\mathcal{M}^\lambda \ni \mu^\lambda \mapsto \widetilde{u}$ constructed above is a linear isomorphism between $\mathcal{M}^\lambda$ and resonant states of X at $i\lambda$.
2. *Finally, the following map is an isomorphism:*
$$\mathcal{J}^\lambda : \mathcal{M}^\lambda \ni \mu^\lambda \mapsto u \in \{w \in L^\infty(\mathcal{M}, \mu) \mid \varphi_t^* w(x) = e^{-it\lambda} w(x), \ \ \forall t \in \mathbb{R}, \ \mu \, \text{a.e.} \, x \in \mathcal{M}\} =: \mathcal{I}^\lambda.$$

Proof. Item 1. Take a basis $\mu_{v_1}^\lambda, \dots, \mu_{v_J}^\lambda$ of $\mathcal{M}^\lambda$ and define the required linear map as $\mathcal{T}(\mu_{v_i}^\lambda) := \widetilde{u}_i$, where $\widetilde{u}_i$ is a resonant state of X at $i\lambda$ constructed in Lemma 9.5 satisfying $\widetilde{u}_i \mu = \mu_{v_i}^\lambda$; extend $\mathcal{T}$ by linearity to $\mathcal{M}^\lambda$. We claim that $(\widetilde{u}_i)_{i=1}^J$ are linearly independent, which would prove the claim as dimensions of the spaces agree by Lemma 9.2. If $\sum_{i=1}^J c_i \widetilde{u}_i \equiv 0$ for some $c_1, \dots, c_J \in \mathbb{C}$, then by summing we get $\sum_{i=1}^J c_i \mu_{v_i}^\lambda \equiv 0$, which implies $c_i \equiv 0$ for each $i = 1, \dots, J$ by assumption and proves the claim. (Note that *a priori* given a resonant state $\widetilde{u}$ we do not know if $\widetilde{u}\mu$ is a measure or if $\widetilde{u}\mu \equiv 0$ implies $\widetilde{u} \equiv 0$, that is, if $\widetilde{u}$ is the unique representative of μ^λ.)

Item 2. For the final isomorphism claim, note that the map is well-defined as $\varphi_t^* \mu^\lambda = e^{-i\lambda t} \mu^\lambda$, so u satisfies the required properties (using $\varphi_t^* \mu = \mu$ and that the density is bounded by Theorem 9.3, Item 3). Moreover, the map is clearly linear and injective. For surjectivity, observe the following: for any $u \in \mathcal{I}^\lambda$, Lemma 9.5 gives a resonant state $\widetilde{u}$ of X at $i\lambda$ such that $u\mu = \widetilde{u}\mu$. Notice that $\widetilde{u}$ is uniquely determined: if $\widetilde{u}_1 \mu = \widetilde{u}_2 \mu$ for resonant states $\widetilde{u}_1$ and $\widetilde{u}_2$ of X at $i\lambda$, then $\widetilde{u}_1 \equiv \widetilde{u}_2$ by the previous paragraph. Again using that $\mathcal{T}$ is an isomorphism, there is a unique $\mu^\lambda \in \mathcal{M}^\lambda$ such that $\mu^\lambda = \widetilde{u}\mu = u\mu$. This completes the proof. □

We will need one more definition:

Definition 9.7. *Let ν be an invariant measure of $(\varphi_t)_{t \in \mathbb{R}}$. Define the* basin *(of* attraction*) to consist of those points $x \in \mathcal{M}$ such that for any $f \in C^0(\mathcal{M})$ we have*
$$\nu(f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi_{-t} x) \, dt.$$
We will say that the measure ν is physical *if the basin has positive volume (i.e. positive Lebesgue measure).*

We are now able to characterise the spectrum on the imaginary axis; let $J \geq 1$ be the dimension of the space of resonant states at zero. Also, we may compute the ergodic decomposition of μ, as well as the basins of attraction, from the resonant states at zero.

Theorem 9.8. *The following properties hold:*

1. *There are precisely J ergodic components of the flow $(\varphi_t)_{t\in\mathbb{R}}$ with respect to the measure μ. More precisely, there are pairwise disjoint, flow invariant sets $(F_i)_{i=1}^J \subset \mathcal{M}$ covering $\mathcal{M}$, which are precisely the ergodic decomposition of μ.*
2. *We have $(\mathbf{1}_{F_i})_{i=1}^J$ is a basis of resonant states of X at zero; by duality $\mu^i := \frac{\mathbf{1}_{F_i}}{\mu(F_i)}\mu$ are linearly independent and span $\mathrm{Res}_{X^*}(0)$. Moreover, for each $i = 1, \dots, J$, the sets F_i satisfy $\mathrm{vol}(F_i) = \mu(F_i)$ and therefore μ^i is a physical measure with F_i its basin of attraction.*
3. *The resonance spectrum of X on the imaginary axis is a union of J additive groups, that is, there are $\lambda_1, \dots, \lambda_J \in \mathbb{R}_{\geq 0}$ such that this spectrum equals*
$$\cup_{i=1}^J i\lambda_i\mathbb{Z}.$$
Moreover, $\dim \mathrm{Res}_X(i\lambda) \leq J$ (see the proof for a more precise statement).
4. *The flow $(\varphi_t)_{t\in\mathbb{R}}$ is mixing with respect to μ if and only if it is ergodic and there are no non-zero resonances on the imaginary axis (that is, $J = 1$ and $\lambda_1 = 0$).*

Proof. Item 1. We start by looking at resonant states at zero. Note that $\mathcal{I}^0$ forms a finite dimensional real algebra (we may restrict to real valued functions in this argument). Note that this algebra does not have any nilpotent elements: if $u^k = 0$ for some $u \in \mathcal{I}^0$ and $k \in \mathbb{N}$, then it follows that $u \equiv 0$. Therefore the Jacobson radical of $\mathcal{I}^0$ is zero and so $\mathcal{I}^0$ is semisimple, so by Artin-Wedderburn's theorem we conclude that $\mathcal{I}^0$ splits as a direct sum of matrix algebras; note that a rank at least two (real or complex) matrix algebra contains nilpotent elements, so each of these ranks equals one. We conclude that there is a basis $(u_i)_{i=1}^J$ of $\mathcal{I}^0$ such that
$$u_i^2 = c_iu_i, c_i \in \mathbb{R}, \quad u_iu_j = 0,\, i \neq j,$$
for some $0 \neq c_i \in \mathbb{R}$ and every $i, j = 1, \dots, J$. If $c_i < 0$ replace u_i by $-u_i$, so we may assume that all the c_i are positive and $u_i(x) \geq 0$ for μ-a.e. $x \in \mathcal{M}$. Moreover, we may assume that $c_i \equiv 1$ by replacing u_i with $\frac{u_i}{c_i}$. Now $u_i^2 = u_i$ so each u_i takes values 1 and 0 (in the measure theoretic sense); thus there are measurable flow-invariant sets $F_1, \dots, F_J \subset \mathcal{M}$ of positive μ measure such that $u_i = \mathbf{1}_{F_i}$ is the indicator function of F_i for each i. Each of the ergodic components F_i is irreducible: otherwise, by Lemma 9.6, this would contradict the fact that u_i are linearly independent and span $\mathcal{I}^0$, proving the first claim.

Item 2. We would like to conclude that $\mathbf{1}_{F_i} \in \mathcal{H}^+$ from $\widetilde{u}_i d\mu \equiv \mathbf{1}_{F_i} d\mu$, but it seems we need some a priori regularity for F_i. By the Birkhoff ergodic theorem, and using the separability of $C^0(\mathcal{M})$, there is a set $\Omega_i \subset F_i$ of full μ-measure such that
$$\mu^i(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi_{-t}^* f(x)\, dt, \quad x \in \Omega_i, \tag{9.19}$$
where $\mu^i := \frac{\mathbf{1}_{F_i}}{\mu(F_i)}\mu$. Note that if $x \in \Omega_i$, then for any $y \in W^{wu}(x)$ we know that the limit in (9.19) is valid (here, $W^u(x) = \{y \in \mathcal{M} \mid d(\varphi_t x, \varphi_t u) \to_{t\to\infty} 0\}$ and $W^{wu}(x) = \cup_{t\in\mathbb{R}} W^u(x)$ denote the *unstable* and *weak unstable* leaves, respectively); denote by $\widetilde{\Omega}_i$ these enlarged sets. Note that $\widetilde{\Omega}_i \cap \widetilde{\Omega}_j = \emptyset$ for $i \neq j$, since otherwise we would get $\mu^i = \mu^j$ by (9.19) which would contradict the disjointness of supports. Using that $\sum_i \mu(F_i) = 1$, we get that $\mu(\widetilde{\Omega}_i) = \mu(F_i)$. Since $\widetilde{\Omega}_i$ are flow-invariant, we may assume that $\widetilde{\Omega}_i = F_i$ for each $i = 1, \dots, J$. Using the absolute continuity of the foliations and the fact that F_i are (measurable) unions of weak unstable leaves, by Lemma 9.13 below we have $\mathrm{WF}(\mathbf{1}_{F_i}) \subset E_u^*$, i.e. $\mathbf{1}_{F_i} \in \mathcal{H}^+$ (for m taking

sufficiently negative values near E_u^*) and so $\mathbf{1}_{F_i} \in \mathrm{Res}_X(0)$. By uniqueness of the resonant state $\widetilde{u}_i$ satisfying $\widetilde{u}_i\mu = \mathbf{1}_{F_i}\mu$ we deduce that $\mathbf{1}_{F_i} = \widetilde{u}_i$ and so $(\mathbf{1}_{F_i})_{i=1}^J \in \mathrm{Res}_X(0)$ forms a basis.

Notice that the defining relation $\mu(u) = \langle \Pi(0)u, \mathbf{1}\rangle$ extends to $u \in \mathcal{H}^+$; therefore using that $\mathbf{1}_{F_i} = \Pi(0)\mathbf{1}_{F_i}$

$$\mu(F_i) = \mu(\mathbf{1}_{F_i}) = \langle \Pi(0)\mathbf{1}_{F_i}, \mathbf{1}\rangle = \int_{F_i} \Omega = \mathrm{vol}(F_i), \quad i = 1, \dots, J.$$

The claim about the physicality of measures immediately follows.

Item 3. Next, notice that $\mathcal{I}^\lambda = \oplus_{i=1}^J \mathcal{I}_i^\lambda$ as a direct sum, where this splitting corresponds to the decomposition $u = \sum_{i=1}^J u\mathbf{1}_{F_i}$. This splitting translates to a splitting on the level of resonant states according to Lemma 9.6. To show the first claim, it suffices to show that for each i, the set of λ for which $\mathcal{I}_i^\lambda \neq \{0\}$ forms an additive subgroup of $\mathbb{R}$.

If there are no such λ, we are done. If there are, take the resonance $i\lambda_i$ with $\lambda_i > 0$ is the smallest possible and consider $0 \not\equiv u \in \mathcal{I}_i^{\lambda_i}$. Now notice that $u^k \not\equiv 0$ in $L^\infty(\mathcal{M}, \mu)$ for any $k \in \mathbb{N}$ (otherwise $u \equiv 0$); thus $0 \not\equiv u^k \in \mathcal{I}_i^{k\lambda_i}$ and again by Lemma 9.6 we conclude that $ik\lambda_i$ is a resonance. To get that $-ik\lambda$ is a resonance we apply the same argument to $\overline{u}$. If there is another $0 < \mu \notin \lambda_i\mathbb{Z}$ and $0 \not\equiv v \in \mathcal{I}_i^\mu$, then using that $0 \neq u^a\overline{v}^b \in \mathcal{I}_{a\lambda - b\mu}^{\lambda_i}$ for any $a, b \in \mathbb{N}$, we obtain a contradiction to the choice of λ. This completes the proof of the first claim.

Next, we claim for each i that $\dim \mathcal{I}_i^\lambda \leq 1$, which would complete the proof of the second item. Assume for the sake of contradiction that there are two linearly independent elements $u_1, u_2 \in \mathcal{I}_i^\lambda$. Then $\overline{u}_1, \overline{u}_2 \in \mathcal{I}_i^{-\lambda}$ and so $|u_1|^2, |u_2|^2, u_1\overline{u}_2 \in \mathcal{I}_i^0$. As $\mathcal{I}_i^0$ is spanned by $\mathbf{1}_{F_i}$, we may assume (after re-scaling) that $|u_1|^2 = |u_2|^2 = \chi_{F_i}$, and also $u_1\overline{u}_2 = c\mathbf{1}_{F_i}$ for some $c \in \mathbb{S}^1$. Since u_1 and u_2 are of modulus 1 on F_i (μ-a.e.), we may invert this to get $u_1 = cu_2$, contradicting the linear independence assumption.

Item 4. This part is almost the same as the volume-preserving case, see Lemma 8.22; we will just spell out the differences. First of all, for $(\varphi_t)_{t\in\mathbb{R}}$ to be ergodic with respect to μ we must have $J = 1$ by Item 1. Moreover, if there is a non-zero resonant state of X at $i\lambda$ for some $\lambda \neq 0$, by Lemma 9.6 there is also $u \in L^\infty(\mathcal{M}, \mu)$ such that $\varphi_t^* u = e^{-it\lambda}u$. Integrating and using that t can be chosen arbitrary we get $\int_{\mathcal{M}} u\, d\mu = 0$. However $\langle \varphi_{-t}^* u, u\rangle = e^{it\lambda}\|u\|_{L^2}^2$ does not converge to zero, so the flow is not mixing.

In the other direction, we have to change the spaces slightly and work with $L^2(\mathcal{M}, \mu)$ and the corresponding domain $\mathcal{D}_{L^2(\mathcal{M},\mu)}(X) = \{u \in L^2(\mathcal{M}, \mu) \mid Xu \in L^2(\mathcal{M}, \mu)\}$ of the (closed) self-adjoint unbounded operator $-iX$ (see Lemma 9.9 below for a proof). The proof of Lemma 8.22 applies verbatim up to the place where we study the pure point spectrum; if $u \in L^2(\mathcal{M}, \mu)$ satisfies $(X + i\lambda)u = 0$ and has mean zero, then Lemma 9.5 applies and gives a non-zero resonant state $\widetilde{u}$ of X at $i\lambda$ such that $\widetilde{u}\mu = u\mu$. By assumption we have $u \equiv 0$ which shows that pure point spectrum is empty; this completes the proof.

□

We finish this section by a classical result that we include for completeness.

Lemma 9.9. *The operator $-iX$ is a closed self-adjoint unbounded operator acting on $\mathcal{D}_{L^2} \subset L^2(\mathcal{M}, \mu)$, where*

$$\mathcal{D}_{L^2} = \{u \in L^2(\mathcal{M}, \mu) \mid Xu := \lim_{t\to 0} \frac{\varphi_t^* u - u}{t} \text{ exists in } L^2(\mathcal{M}, \mu)\}.$$

Proof. Integration by parts gives that $-iX$ is symmetric on $\mathcal{D}_{L^2}$. To see closedness, assume $\mathcal{D}_{L^2} \ni u_i \to u$ and $Xu_i \to v$ in $L^2(\mathcal{M}, \mu)$. Denote by $A_t := \frac{\varphi_t^* - \mathrm{Id}}{t}$ the bounded linear operator acting on $L^2(\mathcal{M}, \mu)$. By integration, the second convergence implies for any t that in $L^2(\mathcal{M}, \mu)$

$$A_t u_i \to \frac{1}{t}\int_0^t \varphi_t^* v\, dt, \quad i \to \infty.$$

Therefore, for any fixed t we have

$$A_t u - \frac{1}{t}\int_0^t \varphi_t^* v\, dt = A_t(u - u_i) + \Big(A_t u_i - \frac{1}{t}\int_0^t \varphi_t^* v\, dt\Big).$$

Since A_t is bounded, both terms on the right hand side converge to zero as $i \to \infty$. Therefore $A_t u \equiv \frac{1}{t}\int_0^t \varphi_t^* v\, dt$. It is left to observe that

$$\|A_t u - v\|_{L^2(\mathcal{M},\mu)} \leq \frac{1}{t}\int_0^t \|\varphi_t^* v - v\|_{L^2(\mathcal{M},\mu)}\, dt \leq \sup_{s\in[0,t]} \|\varphi_t^* v - v\|_{L^2(\mathcal{M},\mu)} = o(1), \quad t \to \infty,$$

since $t \mapsto \varphi_t^* v \in L^2(\mathcal{M}, \mu)$ is continuous, i.e. φ_t^* is a *strongly continuous semigroup*. To see the latter claim, by the one-parameter property and as φ_t^* preserves μ, it suffices to show continuity at $t = 0$; assume $C^\infty(\mathcal{M}) \ni v_i \to v$ in $L^2(\mathcal{M}, \mu)$ as $i \to \infty$. Let $\varepsilon > 0$; then

$$\begin{aligned}\|\varphi_t^* v - v\|_{L^2(\mathcal{M},\mu)} &\leq \|\varphi_t^*(v - v_i)\|_{L^2(\mathcal{M},\mu)} + \|\varphi_t^* v_i - v_i\|_{L^2(\mathcal{M},\mu)} + \|v_i - v_i\|_{L^2(\mathcal{M},\mu)}\\ &= 2\|v - v_i\|_{L^2(\mathcal{M},\mu)} + \|\varphi_t^* v_i - v_i\|_{L^2(\mathcal{M},\mu)}\\ &\leq 2\|v - v_i\|_{L^2(\mathcal{M},\mu)} + \|\varphi_t^* v_i - v_i\|_{C^0} < 3\varepsilon,\end{aligned}$$

where in the second equality we used that φ_t^* preserves μ, in the last line we took i large enough (fixed) so that $\|v - v_i\|_{L^2(\mathcal{M},\mu)} < \varepsilon$ and t arbitrary small enough (depending on i) so that by uniform continuity $\|\varphi_t^* v_i - v_i\|_{C^0} < \varepsilon$. This shows that $t \mapsto \varphi_t^*$ is a strongly continuous semigroup.

Finally, observe that $(-iX \pm i)^{-1} = i(X \mp 1)^{-1} : \mathcal{D}_{L^2} \to L^2(\mathcal{M}, \mu)$ is an isomorphism (where the former space is equipped with the graph norm), thanks to formula (8.18). Since $T := -iX$ is symmetric, it suffices to show that the domain $\mathcal{D}_{L^2}(T^*)$ of T^* is contained in $\mathcal{D}_{L^2}$. Let $u \in \mathcal{D}_{L^2}(T^*)$; then, there exists $v \in \mathcal{D}_{L^2}$ such that $(T^* - i)u = (T - i)v$. Since T is symmetric, we have $Tv = T^*v$ and so $(T^* - i)u = (T^* - i)v$. Thus

$$u - v \in \ker(T^* - i) = \mathrm{ran}(T + i)^\perp = (L^2(\mathcal{M}, \mu))^\perp = \{0\},$$

where in the second equality we used that $T + i$ is surjective. This shows $u = v$ an so $\mathcal{D}_{L^2}(T^*) = \mathcal{D}_{L^2}$ proving that $T = -iX$ is self-adjoint. □

9.2. **Absolute continuity and SRB measures.** We introduce the basic foliation theory behind Anosov flows. We study the absolute continuity of foliations, that is, despite the foliations only being Hölder regular, we show that the holonomy maps are absolutely continuous with respect to smooth measures. We show a similar property for SRB measures studied previously.

We start by considering open neighbourhoods of the basins of physical measures.

Lemma 9.10. *Let μ^j be one of the physical measures constructed in Theorem 9.8. Then, the support of μ^j is a union of full weak stable leaves, and the smallest flow-invariant open neighbourhood U_j of $\mathrm{supp}(\mu^j)$ is given by*

$$U_j := \cup_{x\in\mathrm{supp}(\mu^j)} W^u(x).$$

Moreover, the basin F_j of μ^j is contained and has full Lebesgue measure in U_j. Finally, $\operatorname{supp}(\mu^j)$ *admits an arbitrarily small open neighbourhood that is invariant under $\{\varphi_{-t} \mid t \geq 0\}$, and there is a decomposition*

$$\Pi(0) = \sum_j \mathbf{1}_{U_j} \otimes \mu^j.$$

We derive an easy consequence of this.

Proposition 9.11. *If $(\varphi_t)_{t\in\mathbb{R}}$ is transitive, then $\Pi(0)^*(C^\infty(\mathcal{M}))$ is 1-dimensional and spanned by $\mu = \mu_{\mathbf{1}}$.*

Proof. Assume for the sake of contradiction that there are at least two distinct physical measures μ^1 and μ^2 constructed in Theorem 9.8. By transitivity of the flow there is a $t \in \mathbb{R}$ such that $\emptyset \neq V := \varphi_t(U_1) \cap U_2$, where U_1 and U_2 are open sets defined by Lemma 9.10; by flow-invariance of U_2 we have $V = U_1 \cap U_2$. Since the basins F_i of μ^i have full Lebesgue measure in U_i, we deduce that $F_1 \cap F_2 \neq \emptyset$, which contradicts the fact that $\mu^1 \neq \mu^2$. This completes the proof. $\square$

We emphasise that under the transitivity assumption, Item 4 of Theorem 9.3 and Proposition 9.11 show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_t^* \Omega \, dt = \mu,$$

in the weak limit sense; note that Ω is an *arbitrary* probability volume form.

Next, we show that functions smooth in the weak unstable directions have wavefront set in E_u^*. Before that, we need another definition

Definition 9.12. *Let G be a foliation with smooth leaves, $U \subset \mathcal{M}$ an open set, and $f : U \to \mathbb{C}$ a measurable function. We will say that f is* uniformly smooth *(along the leaves of G) if for almost all $y \in U$ we have $f|_{G_{\mathrm{loc}}(y)} \in C^\infty(G_{\mathrm{loc}}(y))$ and if for all $k \in \mathbb{N}$*

$$\operatorname{ess\,sup}_{y\in U} \|f|_{G_{\mathrm{loc}}(y)}\|_{C^k(G_{\mathrm{loc}}(y))} < \infty.$$

It is then possible to show:

Lemma 9.13. *Let f be a measurable function on $\mathcal{M}$ which is uniformly smooth along the leaves of the weak unstable foliation W^{wu}. Then $\mathrm{WF}(f) \subset E_u^*$.*

References

[1] S. Dyatlov, M. Zworski, *Mathematical theory of scattering resonances.* Graduate Studies in Mathematics, 200. American Mathematical Society, Providence, RI, 2019. xi+634 pp.

[2] F. Faure, J. Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. in Math. Physics, vol. 308 (2011), 325-364.

[3] F.G. Friedlander, *Introduction to the theory of distributions*, Second edition. With additional material by M. Joshi, Cambridge University Press, Cambridge, 1998.

[4] A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators. An introduction*, London Mathematical Society Lecture Note Series, 196. Cambridge University Press, Cambridge, 1994.

[5] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Translated from the 1978 Russian original by Stig I. Andersson, Second edition, Springer-Verlag, Berlin, 2001.

[6] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI, 2012.